Math 210 Midterm 2 Answers

Problem 1 (5 pts) *Find the derivative:*

$$f(x) = x^5 + 10x^3 + 5x + 8$$

$$f'(x) = 5x^4 + 10 \cdot 3x^2 + 5$$

Problem 2 (5 pts) *Find the derivative:*

$$g(t) = 10e^{8t}$$

Constant multiple rule and chain rule.

$$g'(t) = 10e^{8t} \cdot 8$$

Problem 3 (5 pts) *Find the derivative:*

$$S(x) = x^8 \tan x$$

Product rule.

$$S'(x) = 8x^7 \tan x + x^8 \sec^2 x$$

Problem 4 (5 pts) *Find the derivative:*

$$Q(x) = \sin^{-1}(4x^3)$$

Use the chain rule.

$$Q'(x) = \frac{1}{\sqrt{1-(4x^3)^2}} 4 \cdot 3x^2$$

Problem 5 (5 pts) *Find the derivative:*

$$W(u) = \frac{3u+1}{5+\sqrt{7u+2}}$$

Use the quotient rule for a quotient.

$$W'(u) = \frac{(3)(5+\sqrt{7u+2}) - (3u+1)(0+\frac{1}{2}(7u+2)^{-1/2} \cdot 7)}{(5+\sqrt{7u+2})^2}$$

Problem 6 (5 pts) *Find the derivative, assuming b is a constant:*

$$f(x) = 3x^5 b + \ln b$$

Remember that $\ln b$ is a constant, and use the sum rule.

$$f'(x) = 3 \cdot 5x^4 b + 0$$

Problem 7 (5 pts) *Find the derivative:*

$$f(x) = 2^{\left(3^{(5^x)}\right)}$$

Apply the chain rule many times:

$$\ln 2 \cdot 2^{\left(3^{(5^x)}\right)} \ln 3 \cdot 3^{(5^x)} \ln 5 \cdot 5^x$$

Problem 8 (5 pts) *Find the second derivative of*

$$x^2 \sin x$$

First derivative:

$$2x \sin x + x^2 \cos x$$

Second derivative:

$$2 \sin x + 2x \cos x + 2x \cos x + x^2 (-\sin x)$$

Problem 9 (5 pts) *Below is a table for f, g, f', and g'. If $h(x) = f(x)g(x)$, find $h'(2)$.*

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	4	7	8	12
1	2	2	2	20
2	3	4	2	10
3	1	9	3	11
4	0	1	6	3

Using the product rule,

$$\begin{aligned} h'(2) &= f'(2)g(2) + f(2)g'(2) \\ &= 2 \cdot 4 + 3 \cdot 10 \\ &= 8 + 30 = 38 \end{aligned}$$

Problem 10 (5 pts) *For the above table, suppose $W(x) = f(g(x))$. Find $W'(2)$.*

Using the chain rule,

$$\begin{aligned} W'(2) &= f'(g(2)) \cdot g'(2) \\ &= f'(4)g'(2) \\ &= 6 \cdot 10 = 60 \end{aligned}$$

Problem 11 (5 pts) *Use logarithmic differentiation to find the derivative of f:*

$$f(x) = \frac{x^3 e^x \ln x}{(5x+3)\sin x}$$

Taking the log of both sides,

$$\begin{aligned}\ln f(x) &= \ln\left(\frac{x^3e^x\ln x}{(5x+3)\sin x}\right)\\ &= (\ln x^3+\ln e^x+\ln\ln x)-(\ln(5x+3)+\ln\sin x)\\ &= 3\ln x+x+\ln\ln x-\ln(5x+3)-\ln\sin x\end{aligned}$$

Then take the derivative of both sides:

$$\frac{1}{f(x)}f'(x) = \frac{3}{x}+1+\frac{1}{\ln x}\frac{1}{x}-\frac{1}{5x+3}\cdot 5-\frac{1}{\sin x}\cdot\cos x$$

And multiply both sides by $f(x)$:

$$f'(x) = \left(\frac{3}{x}+1+\frac{1}{\ln x}\frac{1}{x}-\frac{5}{5x+3}-\frac{\cos x}{\sin x}\right)\left(\frac{x^3e^x\ln x}{(5x+3)\sin x}\right)$$

Problem 12 (5 pts) *True or False:*

FALSE *Wherever $f'(x)$ is increasing, $f(x)$ is also increasing.* **No, when $f'(x)$ is positive, that is when $f(x)$ is increasing.**

TRUE *If $f(x)$ is differentiable at $x=2$, then it is continuous at $x=2$.* **Yes, to be differentiable, the function must first be continuous.**

FALSE *If $f'(x)=x^x$ and $g'(x)=\ln x$, and $h(x)=f(x)g(x)$, then the derivative of $h(x)$ is $x^x\ln x$.* **No, to find $h'(x)$ we must use the product rule.**

TRUE *When h is close to 0, then $\frac{e^h-1}{h}$ is close to 1.* **Yes, this is how e is defined at first when we found the derivative of a^x.**

FALSE *No matter the value of x, $\sin^{-1}(x)$ is always between -1 and 1 (inclusively).* **It is $\sin(x)$ that is always between -1 and 1. The inverse sine function is always between $-\pi/2$ and $\pi/2$, which is a somewhat bigger interval.**

Problem 13 (5 pts) *Find the slope of the tangent line to the function $y(x)$ given by the equation below, at the point (1,2).*

$$x^2y-y^3+7=1$$

We do implicit differentiation:

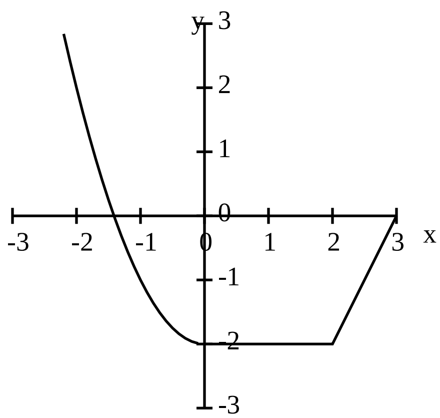
$$\begin{aligned}\frac{d}{dx}(x^2y-y^3+7) &= \frac{d}{dx}(1)\\ 2xy+x^2y'-3y^2y'+0 &= 0\\ (x^2-3y^2)y' &= -2xy\\ y' &= \frac{-2xy}{x^2-3y^2}\end{aligned}$$

We now plug in (1,2) and get

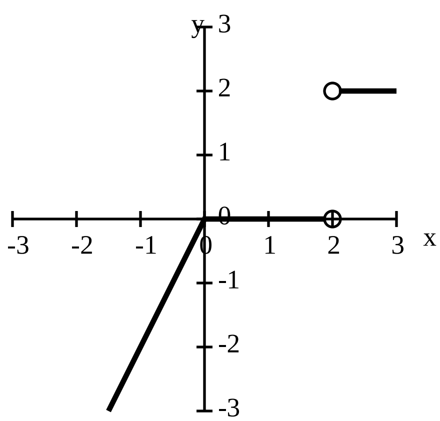
$$y' = \frac{-2 \cdot 1 \cdot 2}{1^2 - 3 \cdot 2^2} = \frac{-4}{-11} = \frac{4}{11}$$

Problem 14 (5 pts) *Below is a graph of $f(x)$. Sketch a graph of $f'(x)$.*

$f(x)$:



$f'(x)$:



Note that f is decreasing when $x < 0$, but concave up. This means f' is negative but increasing. It increases to 0, the slope of f at $x = 0$. From $x = 0$ to $x = 2$ the slope is 0 so $f' = 0$. From $x = 2$ to $x = 3$ the slope is about 2, so f' is about 2, staying constant throughout. Also, at $x = 2$, there is a corner, so the $f'(2)$ is not defined. I drew open circles in the answer to make that clear.

Problem 15 (5 pts) *Prove that*

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

We start with

$$y = \ln x$$

and use the fact that ln and e^x are inverses to get

$$e^y = x$$

We implicitly differentiate:

$$\begin{aligned}
\frac{d}{dx}(e^y) &= \frac{d}{dx}(x) \\
e^y \frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{e^y} \\
&= \frac{1}{x}
\end{aligned}$$

where in the last step we used $e^y = x$.

Problem 16 (5 pts) *Prove that*

$$\frac{d}{dx}(f(x) + g(x)) = \frac{df}{dx} + \frac{dg}{dx}$$

$$\begin{aligned}
\frac{d}{dx}(f(x) + g(x)) &= \lim_{h\to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right) \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \frac{df}{dx} + \frac{dg}{dx}
\end{aligned}$$

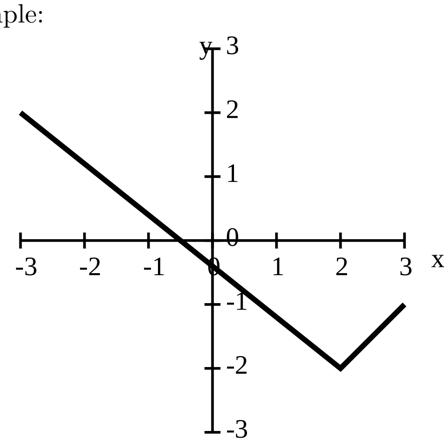
Problem 17 (5 pts) *Prove that*

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\begin{aligned}
\frac{d}{dx}\sec x &= \frac{d}{dx}\frac{1}{\cos x} \\
&= \frac{0\cdot\cos x - 1\cdot(-\sin x)}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \frac{1}{\cos x}\frac{\sin x}{\cos x} \\
&= \sec x \tan x
\end{aligned}$$

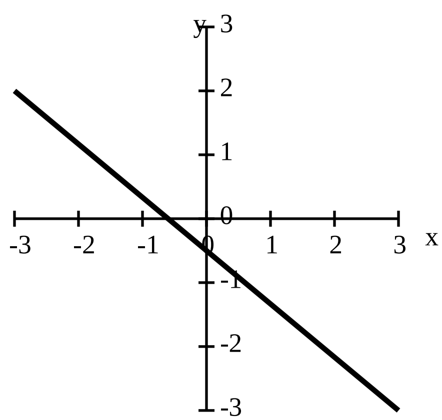
Problem 18 (5 pts) *Draw an example of a graph of a function that is defined everywhere but not differentiable at $x = 2$.*

Anything with a corner, a vertical tangent line, or a discontinuity at $x = 2$ is fine. As long as it is defined for all x. Here is a corner, for example:



Problem 19 (5 pts) *Draw an example of a graph of a function $f(x)$ so that $f'(x)$ is always negative.*

Anything decreasing, like this:



Problem 20 (5 pts) *Suppose a function f has been found to satisfy the equation*

$$f'(x) = f(x) + \sin f(x)$$

Find the derivative of the inverse of f.

If we have

$$y = f^{-1}(x)$$

then

$$f(y) = x$$

and take the derivative of both sides:

$$\begin{aligned} \frac{d}{dx} f(y) &= \frac{d}{dx} x \\ f'(y) \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{f'(y)} \end{aligned}$$

Now we use the fact given in the problem to write

$$\frac{dy}{dx} = \frac{1}{f(y) + \sin f(y)}$$

and use the fact that $f(y) = x$ to write

$$\frac{dy}{dx} = \frac{1}{x + \sin x}$$