Exercises

The systems in Exercises 1 – 2 are to be solved using Jacobi's Method with the following details:

- Using $\boldsymbol{x}^{(0)} = \boldsymbol{0}$, complete the table below, doing five iterations:

k	$\boldsymbol{x}^{(k)}$		$\boldsymbol{x}^{true} - \boldsymbol{x}^{(k)}$		$\lVert error^{(k)} \rVert$
1	4.000000	-1.250000	-1.000000	-0.750000	1.250000
2					
3					
4					
5					

- Compute the first two iterations $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ by hand (show your work!), and use the applet to perform the next three iterations.
- Do more iterations (don't write down the details of the results; that is, don't make a table for these next iterations) until $\lVert error^{(k)} \rVert \le 0.000100$. How many iterations are required?

1.
$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & 8 \\ x_1 & + & 4x_2 & = & -5 \end{array}$$

2.
$$\begin{array}{rcrcr} 10x_1 & + & 9x_2 & = & 19 \\ 8x_1 & + & 11x_2 & = & 19 \end{array}$$

3. Do Exercise 1, using the Gauss-Seidel Method instead of Jacobi's Method.

4. Do Exercise 2, using the Gauss-Seidel Method instead of Jacobi's Method.

5. The following two systems

(A)
$$\begin{array}{rcrcr} 11x_1 & + & 3x_2 & = & 26 \\ 3x_1 & + & 12x_2 & = & 63 \end{array}$$

(B)
$$\begin{array}{rcrcr} 11x_1 & + & 10x_2 & = & 61 \\ 10x_1 & + & 12x_2 & = & 70 \end{array}$$

happen to have the same solution. Solve each using the following details:

- Solve the system using both the Jacobi and Gauss-Seidel Methods. Using each method, solve the system three different times, each time using a different initial guess: $\boldsymbol{x}^{(0)} = (0, 0)$, $\boldsymbol{x}^{(0)} = (100, -50)$, and $\boldsymbol{x}^{(0)} = (-500, 1000)$ (so in all you will have six sets of results).
- Use the applet to compute all iterations, and iterate until the applet shows $\lVert error^{(k)} \rVert$ as 0.000000 (the error is not really exactly 0, but it rounds to 0 to six digits after the decimal). How many iterations were required for each method when using the three different initial guesses? Complete the table given below, in which you compare the results for each of the six cases (the results of the first case are given).

System	Initial Guess	# iterations until $\|error^{(k)}\| = 0$		Jacobi / Gauss-Seidel
		Jacobi	Gauss-Seidel	
A	$(0,0)$	13	7	$13/7 \approx 1.857$
A	$(100,-50)$			
A	$(-500,1000)$			
B	$(0,0)$			
B	$(100,-50)$			
B	$(-500,1000)$			

What is the relationship (approximately) between the number of iterations required for the Jacobi Method and the number of iterations required for the Gauss-Seidel Method? Suppose you were told that for a certain system of equations and given a certain initial guess, the Gauss-Seidel Method required $k = 100$ iterations to get to $\|error^{(k)}\| = 0$. About how many iterations would the Jacobi Method would require to get to $\|error^{(k)}\| = 0$?

6. We know that an iterative method, such as the Jacobi or Gauss-Seidel Methods, will (hopefully) produce a sequence of approximations that get closer and closer to the true solution. In this problem we consider the question of whether or not we ever reach the true solution exactly. You will use Jacobi's Method to solve the system

$$\begin{aligned} 3x_1 &+ 2x_2 &= 5 \\ x_1 &+ 4x_2 &= 5 \end{aligned}.$$

Since the true solution is $\mathbf{x} = (1,1)$, let us center the viewing window around that point, by changing the minimum and maximum boundaries for both x_1 and x_2 to –4 and 6. For this problem use an initial guess of $\boldsymbol{x}^{(0)} = (5,-5)$. Also, for this problem you will not write down the results of your iterations.

(a) Do 10 iterations. On the graph in the applet, does it appear that the approximations have already reached the true solution? Now zoom in about 10 times by clicking on the ***Zoom in*** button, and answer the same question.

(b) Do 10 more iterations, for a total of 20, and answer the same question as in (a). As done in (a), zoom in about 10 more times and answer the same question again.

(c) Does it appear that we will ever reach the solution exactly? Although it would be nice to have the true solution exactly, is an approximation actually good enough? (Note: if you attempt to continue to iterate and zoom in, you will eventually, perhaps quickly, exhaust the precision of your computer, and it may produce strange results—your computer can zoom in only so far.)

7. For the system of equations

$$\begin{aligned} x_1 &+ 2x_2 &= 1 \\ 2x_1 &+ x_2 &= 2 \end{aligned},$$

use the applet to compute five iterations when using both the Jacobi and Gauss-Seidel Methods, using initial value $\boldsymbol{x}^{(0)} = \boldsymbol{0}$, and record your results. For each method, make and complete the following table.

k	$\boldsymbol{x}^{(k)}$		$\| error^{(k)} \|$
1	1.000000	2.000000	2.000000
2			
3			
4			
5			

Also, find the eigenvalues of the B matrix when solving the system using each method, and use this information to explain why the approximations are not converging for either method.

Now rearrange (swap) the two equations so that we have

$$\begin{aligned} 2x_1 &+ x_2 &= 2 \\ x_1 &+ 2x_2 &= 1 \end{aligned},$$

and use the applet to compute five iterations, again making a table for each method. This time the approximations produced by the iterations should be converging. Find the eigenvalues of the B matrix when solving the system using each method, and use this information to explain why the approximations are converging in this case. Note: this shows you that the order of the equations can affect the performance of a method.

8. Show that if A is a strictly diagonally dominant 2 x 2 matrix, then the eigenvalues of the B matrices corresponding to the Jacobi and Gauss-Seidel Methods are of magnitude < 1 (which guarantees convergence of the Jacobi Method in finding the solution to $A\boldsymbol{x} = \boldsymbol{b}$, as subsequently shown in Exercise 9). Make up a system of equations in which the matrix A is strictly diagonally dominant (it doesn't matter what $\boldsymbol{b}$ is nor what the initial guess $\boldsymbol{x}^{(0)}$ is), and compute the first five iterations (make a table) to help verify that both methods produce approximations that are converging to the exact solution.

9. Suppose that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent eigenvectors of B, and suppose that $\boldsymbol{e}^{(0)} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2$. Recall that the *error* at iteration k is $\boldsymbol{e}^{(k)} = B\boldsymbol{e}^{(k-1)}$, so that

$$\boldsymbol{e}^{(1)} = B\boldsymbol{e}^{(0)} = B(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 .$$

(a) Prove (by induction) that $\boldsymbol{e}^{(k)} = c_1\lambda_1^k\boldsymbol{v}_1 + c_2\lambda_2^k\boldsymbol{v}_2$.

(b) Since $\left\|\boldsymbol{e}^{(k)}\right\| = \left\|c_1\lambda_1^k\boldsymbol{v}_1 + c_2\lambda_2^k\boldsymbol{v}_2\right\| \le \left\|c_1\lambda_1^k\boldsymbol{v}_1\right\| + \left\|c_2\lambda_2^k\boldsymbol{v}_2\right\| = |c_1|\,|\lambda_1|^k\left\|\boldsymbol{v}_1\right\| + |c_2|\,|\lambda_2|^k\left\|\boldsymbol{v}_2\right\|$, show that if $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then $\left\|\boldsymbol{e}^{(k)}\right\| \to 0$ as $k \to \infty$.

10. Consider the Jacobi's Method for solving the system $A\boldsymbol{x} = \boldsymbol{b}$, where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

(a) Verify that the B matrix corresponding to the Jacobi Method for this system is $B = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$, and verify that the eigenvalues for this matrix are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{2}$ with corresponding eigenvectors $\boldsymbol{v}_1 = (1,-1)$ and $\boldsymbol{v}_2 = (1,1)$ (simply show that $B\boldsymbol{v}_1 = \lambda_1\boldsymbol{v}_1$ and $B\boldsymbol{v}_2 = \lambda_2\boldsymbol{v}_2$).

(b) Suppose that $\boldsymbol{b} = (0,0)$ and $\boldsymbol{x}^{(0)} = (5,-1)$. Since for $\boldsymbol{b} = (0,0)$ the true solution is $\boldsymbol{x} = (0,0)$, then $\boldsymbol{e}^{(0)} = \boldsymbol{x} - \boldsymbol{x}^{(0)} = (-5,1)$. Verify that $\boldsymbol{e}^{(0)} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2$, where $c_1 = -3$ and $c_2 = -2$, and where $\boldsymbol{v}_1 = (1,-1)$ and $\boldsymbol{v}_2 = (1,1)$ are the eigenvectors of B.

(c) Use the fact that $\boldsymbol{e}^{(k)} = c_1\lambda_1^k\boldsymbol{v}_1 + c_2\lambda_2^k\boldsymbol{v}_2$ to show $\boldsymbol{e}^{(k)} = \begin{cases} (\frac{-5}{2^k}, \frac{1}{2^k}) & \text{for } k \text{ even} \\ (\frac{-1}{2^k}, \frac{5}{2^k}) & \text{for } k \text{ odd} \end{cases}$.

Compute the errors $\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}, \boldsymbol{e}^{(3)}, \boldsymbol{e}^{(4)}, \boldsymbol{e}^{(5)}$ when doing five iterations of the Jacobi Method using $\boldsymbol{x}^{(0)} = (5,-1)$, as given in (b). Show your work, and also verify that your work is correct by using the applet to also compute these errors.

(d) Now suppose that $\boldsymbol{b} = (3,0)$ so that the true solution is $\boldsymbol{x} = (2,-1)$. Suppose that $\boldsymbol{x}^{(0)} = (7,-2)$, so that $\boldsymbol{e}^{(0)} = \boldsymbol{x} - \boldsymbol{x}^{(0)} = (-5,1)$, as in (b)/(c). Repeat (b) and (c) and find the errors $\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}, \boldsymbol{e}^{(3)}, \boldsymbol{e}^{(4)}, \boldsymbol{e}^{(5)}$. How to the errors in this case compare to the errors found in (b)/(c)?

(e) Now suppose that any $\boldsymbol{b}$ and $\boldsymbol{x}^{(0)}$ are chosen such that the initial error is $\boldsymbol{e}^{(0)} = \boldsymbol{x} - \boldsymbol{x}^{(0)} = (-5,1)$, as was the case in (b) – (d). What would you predict the errors $\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}, \boldsymbol{e}^{(3)}, \boldsymbol{e}^{(4)}, \boldsymbol{e}^{(5)}$ would be?

(f) Support your answer in (f) by finding any $\boldsymbol{b}$ and $\boldsymbol{x}^{(0)}$ such that $\boldsymbol{e}^{(0)} = (-5,1)$ (you choose $\boldsymbol{b}$ and $\boldsymbol{x}^{(0)}$; there are an infinite number of possibilities), and then use the applet to compute the errors $\boldsymbol{e}^{(1)}, \boldsymbol{e}^{(2)}, \boldsymbol{e}^{(3)}, \boldsymbol{e}^{(4)}, \boldsymbol{e}^{(5)}$. Hint: first choose $\boldsymbol{x}$ (which will give you $\boldsymbol{b}$, since $A\boldsymbol{x} = \boldsymbol{b}$), and then choose $\boldsymbol{x}^{(0)}$ such that $\boldsymbol{e}^{(0)} = \boldsymbol{x} - \boldsymbol{x}^{(0)} = (-5,1)$; that is, $\boldsymbol{x}^{(0)} = \boldsymbol{x} - (-5,1)$.

11. Do Exercise 10, using the Gauss-Seidel Method instead of the Jacobi Method. You will need to find B and its eigenvalues and eigenvectors, as well as c_1 and c_2.

12. Consider solving each of the following systems using Jacobi's Method:

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

(a) Each of these has the same solution, but the convergence properties for each are quite different. For each system, use the applet to do five iterations (the initial guess won't really matter, so simply use $\boldsymbol{x}^{(0)} = \boldsymbol{0}$). Record the $\| error \|$ at each iteration, and describe what the graph in the applet is showing.

(b) For each of the three systems, find the B matrix and the corresponding eigenvalues, and use these to explain the results in (a). Note: the "circular" behavior of the Jacobi Method when applied to the second of the three systems occurs because the eigenvalues of the B matrix are *complex*.

13. Do Exercise 12, using the Gauss-Seidel Method instead of Jacobi's Method.

14. For this problem we consider the convergence rate of Jacobi's Method when solving

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

Recall that $\| \boldsymbol{e}^{(k)} \| = \| B\boldsymbol{e}^{(k-1)} \| \approx | \lambda_{\max} | \, \| \boldsymbol{e}^{(k-1)} \|$, where $\lambda_{\max}$ is the largest (in magnitude) eigenvalue of the B matrix. This approximate relationship often is not the case for the first few iterations, but the approximation often becomes more valid for later iterations. It turns out that for 2 x 2 systems, if both eigenvalues are equal then we will have $\| \boldsymbol{e}^{(k)} \| = | \lambda_{\max} | \, \| \boldsymbol{e}^{(k-1)} \|$ exactly and immediately, which also means that $\| \boldsymbol{e}^{(k)} \| \,/\, \| \boldsymbol{e}^{(k-1)} \| = | \lambda_{\max} |$.

(a) Find the two eigenvalues of the B matrix corresponding to Jacobi's Method when solving the above system.

(b) Verify the relationship $\| \boldsymbol{e}^{(k)} \| \,/\, \| \boldsymbol{e}^{(k-1)} \| = | \lambda_{\max} |$ by completing the following table and then comparing these results to the eigenvalues found in (a). Use $\boldsymbol{x}^{(0)} = \boldsymbol{0}$.

k	$\| \boldsymbol{e}^{(k)} \|$	$\| \boldsymbol{e}^{(k)} \| \,/\, \| \boldsymbol{e}^{(k-1)} \|$
1	0.942809	–
2	0.628539	0.666667 $\longleftarrow$ 0.628539 / 0.942809
3		
4		

(c) In general, what would $\dfrac{\| \boldsymbol{e}^{(k)} \|}{\| \boldsymbol{e}^{(k-1)} \|}$ be for $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$?

15. Do Exercise 14, using the Gauss-Seidel Method instead of Jacobi's Method.

16. Consider the SOR Method for the 2 x 2 system $\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

(a) Use the SOR Method to solve the system with the following details:

- Generally we choose a value for ω where $1 < \omega < 2$. For each value of $\omega = 1.1,\ 1.2,\ 1.3,\ 1.4,\ 1.5,\ 1.6,\ 1.7,\ 1.8$ and 1.9, solve the system using $\boldsymbol{x}^{(0)} = (0,0)$, and record the $\| error^{(k)} \|$ after $k = 10$ iterations. Use the applet to perform the iterations. (Be sure to hit the "Initial guess" button after entering each new value of ω.) You do not need to write down the results at each step—this time we are interested only in the final result when using each value of ω.
- Using your nine errors (for each of the nine values of ω), decide which value of ω seems to result in the fastest convergence (which one gave the smallest error after 10 iterations?). Note: this is not necessarily the optimal value of ω; it is simply the best choice of ω from the nine values that we considered.

(b) Repeat (a), except use $\boldsymbol{x}^{(0)} = (-20,10)$ as your initial guess.

(c) Explain why the best value of ω is the same in both (a) and (b).

(d) For each initial guess $\boldsymbol{x}^{(0)} = (0,0)$ and $\boldsymbol{x}^{(0)} = (-20,10)$, solve the system using the Gauss-Seidel Method (which is equivalent to the SOR Method where $\omega = 1$) and for both cases record the $\| error \|$ after 10 iterations. For which values of ω given in part (a) does the SOR Method produce faster convergence results (that is, a smaller $\| error \|$ after 10 iterations) than the Gauss-Seidel Method and for which values is the convergence worse?

17. Consider the SOR Method for solving the 2 x 2 system $\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

(a) Find the B matrix when using the SOR Method for this system (as a function of ω).

(b) The characteristic equation for this B matrix is

$$\det(B - \lambda I) = \tfrac{1}{16}\left[\lambda^2 - (9\omega^2 - 32\omega + 32)\lambda + 256(\omega - 1)^2\right]$$

One could use the quadratic formula to find the two eigenvalues (as functions of ω) that satisfy $\det(B - \lambda I) = 0$. It turns out when the two eigenvalues are the same, then $\| B \| = | \lambda_{\max} |$ is minimized, which results in the fastest possible convergence for the SOR Method. The two eigenvalues will be the same when in $a\lambda^2 + b\lambda + c$ the discriminant $b^2 - 4ac = 0$, where $a = 1$, $b = -(9\omega^2 - 32\omega + 32)$, and $c = 256(\omega - 1)^2$. That is, we want to find where $b^2 = 4ac$, that is,

$$\begin{aligned} [-(9\omega^2 - 32\omega + 32)]^2 &= 4\,[\,256(\omega - 1)^2] \\ \Rightarrow \quad 9\omega^2 - 32\omega + 32 &= 2\sqrt{256(\omega-1)^2} \\ \Rightarrow \quad 9\omega^2 - 32\omega + 32 &= 32\,(\omega - 1) \end{aligned}$$

which we can simplify to $9\omega^2 - 64\omega + 64 = 0$.

Find the value of ω that results in the two eigenvalues being the same by solving for ω in $9\omega^2 - 64\omega + 64 = 0$ (we want the value that is $1 < \omega < 2$).

(c) If Exercise 16 was assigned, compare the value you found in (b) to the results you found in Exercise 16 for an optimal value of ω (of those given in 16(a)).

18. *Warning: this problem is for the determined, patient and strong-hearted student.* Consider the SOR Method for solving the general 2 x 2 system

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

(a) Find the *B* matrix and its eigenvalues (as functions of $a_{11}, a_{12}, a_{21}, a_{22}$ and ω).

(b) Find the value of ω which results in the two eigenvalues being the same (this is the value of ω which minimizes $\|B\| = |\lambda_{\max}|$). This ω is a function of $a_{11}, a_{12}, a_{21}, a_{22}$. What is the single eigenvalue (with algebraic multiplicity 2) in this case?

(c) For the ω found in (b), what conditions on $a_{11}, a_{12}, a_{21}, a_{22}$ guarantee that $\|B\| = |\lambda_{\max}| < 1$?