Very preliminary version

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1 Introduction

1.1 What is Jeopardy?

Jeopardy[1], one of the most popular game shows on TV, is a contest between three players, hosted by the emcee Alex Trebek. The show is divided into three rounds: Single Jeopardy, Double Jeopardy, and Final Jeopardy. Play begins in Single Jeopardy with the contestants giving answers in the form of a question to various challenges posed by Alex Trebek. Each challenge has a monetary value between \$100 and \$500, which a player may gain or lose, depending upon whether they give a correct answer or not. After Single Jeopardy comes Double Jeopardy, which is similar to Single Jeopardy, but the monetary value of the challenges is doubled. The players continue to accumulate money until the end of this round, and the game then moves to Final Jeopardy, which is the round of interest here.

Of the three starting players, the ones with positive dollar amounts at the end of Double Jeopardy are allowed to participate in Final Jeopardy. At the beginning of this round, the players are given the subject of one last answer for which they will try to provide the question. Each player then decides simultaneously and in secret how much of the money they have accumulated they will wager on this final challenge. The challenge is then revealed, and each player tries to provide the correct question to the answer. The players who are correct have their wagers added to the total, and those who are incorrect have their wagers deducted. The game then ends, and the winner is determined by the player with the most money at the end of Final Jeopardy. The winner retains the day's winnings and is allowed to return the next day to try to win more money, while the losers receive lovely parting gifts.

Here we shall try to determine the optimal strategies for Final Jeopardy when one of the three players is unable to participate (that is, enters Final Jeopardy with $\leq \$0$).

1.2 Notation

There are 2 opponents in this game; players I and II, who will be referred to as Alice, and Bob, respectively. The players:

- each have a certain amount of money to begin with. By default, Alice shall be the player with more money at the beginning of the game.
- each make a bet; b_1, b_2, respectively, which may be any amount from \$0 to their total at the beginning of the game. These wagers are made simultaneously, and the players do not consult one another about their bets.

  However, since the players have so many choices for their bets, we will be dealing with matrix games on the order of 1000x1000, and easily even bigger. Therefore, we shall normalize the players' starting amounts to 1 for Alice, and $m = \frac{Bob's starting money}{Alice's starting money}$ for Bob. You may object, of course, that we have violated the game by allowing continuous rather than discrete strategies, but because of the immense number of possible strategies, we can do this without losing the essential spirit of the game.

[1]The game is correctly known as Jeopardy!, but to avoid confusion with punctuation, the exclamation point will be omitted.

The bets are also normalized, and we can represent the players' strategies with a 2-dimensional box with axes to represent each player's range of possible bets. Each point in that box represents a different set of bets from the players, and has a payoff attached to it.

Also note that, to represent smaller dollar amounts, $\epsilon = \$1$.

- have certain probabilities of answering the challenge correctly, which shall be denoted by p_{00}=a=Pr(neither is correct), p_{01}=b=Pr(Alice is wrong and Bob is right), and p_{11}=c=Pr(both are correct). Note that p_{10}=1-a-b-c, and is a guaranteed win for Alice.

1.3 Rule Changes and Necessary Assumptions

Payoffs are different from game show
The winner gets 1, the loser gets zero. The emphasis here is on coming back the next day, since the winner will have the chance to win money the next day, and today's losers will still only have their parting gifts.

Game is now zero-sum (coin flip in case of tie)
In case of a tie on Jeopardy, both players "win". Here, however, each player receives 1/2 when a tie is reached at the end of the game. This represents the fact that returning to play again is mitigated by the fact that one of your opponents is assumedly as good as you are.

Players know each other's probabilities
Each player must be able to make some reasonably accurate guess of their own and the other players' chances of getting the Final Jeopardy question right. This is because the payoffs are determined in a large part by who gets the question right and who gets it wrong. So long as the payoff function is continuous, this assumption is fine.

Players are intelligent and play optimally
We shall assume that all the contestants are there to win money and not simply play for the fun of being on national TV; and also that they are not ignorant of optimal betting strategies, so they will not make unintelligent bets.

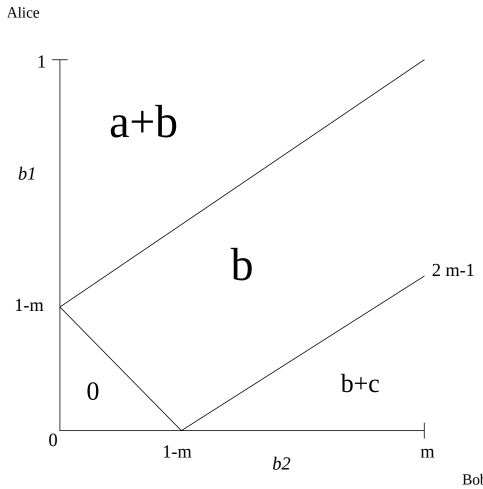

Figure 1: The 2-person graph

Players play to come back next day, not to maximize their winnings
This is linked to the decision to make the payoffs 1 for the winner and 0 for the losers. The assumption inherent here is that the possibility of winning a lot of money the next day outweighs the possible extra money one could make today by risking everything, which can be an especially dangerous bet for the player in the lead. The important result that this assumption gives us is that Alice will **never** bet more than $2m-1+\epsilon$; that is, never more than just enough to beat Bob if Bob should bet everything. This statement of dominance over all betting strategies greater than $2m-1+\epsilon$ will begin the search for solutions in all the cases we will examine.

2 The 2-person game

2.1 Putting it together

The game graph has four plateaus, representing the payoff. This graph is a continuous representation of the discrete game, and it is by using this graph that we will solve the 2-person game.

The uppermost section of the graph is comprised of combinations of bets for which Bob wins if Alice is wrong. Bob wins in the middle section only if he is right and Alice is wrong, he never wins in the lower left section, and wins in the lower right if he is right. The payoffs for the lines are the averages of the payoffs for the regions they lie between, and the payoffs for the corners are the averages of the payoffs for the regions along the axis they lie between.

It is not so obvious who will win, or what should be wagered. This is complicated by the fact that the graph changes for different m. This difficulty can be overcome if we consider the map for various values of m separately, rather than attempt to solve them all at once. We will find that though the graph changes for different m, the solutions are similar for wide ranges of that variable.

2.2 solution

So, let's think about which ranges of m we'd like to consider first. The first one to pop into mind is $m \in [0, \frac{1}{2})$, the values of m for which Alice can't lose (as long as she bets intelligently, which we have assumed). Then, we might consider $m = \frac{1}{2}$, because we have made the game zero-sum, so it is possible for Alice to lose there.

2.2.1 $m = \frac{1}{2}$

Here we have the first interesting case. Assuming that Alice will not bet more than just enough to exceed Bob's maximum possible final money, or $2m - 1 + \epsilon$; which we have, we see that Alice's set of strategies is to bet \$1 or nothing, since $2m - 1 + \epsilon$ is \$1 in this case. Then Bob has a dominant strategy, namely, to bet everything.

In order to avoid wordy explanations and reader confusion in the remaining cases, they shall be explained in the following manner:

I) $2m - 1 + \epsilon$ dominates $[\ 2m - 1 + \epsilon, 1\ ]$

II) m dominates all of Bob's other strategies.

This leaves Alice with the strategies: $0, 2m-1+\epsilon =$ \$1,

which gives us a 1x2 matrix game:

$$\begin{bmatrix} \frac{b+c}{2} & b \end{bmatrix}$$

And, if we solve this matrix game, we find that Alice will bet \$1 only if $b > c$

Next we might consider $m \in (1/2, 2/3)$, because at $m = 2/3$ Bob acquires an optimal strategy other than betting everything, since should Alice bet $2m-1+\epsilon = 1/3 + \epsilon$ and miss the question, her final amount will be less than Bob's starting amount.

2.2.2 $m \in (1/2, 2/3)$

The interesting thing about this case is that each player has a single dominant strategy.

I) $2m - 1 + \epsilon$ dominates $[2m - 1 + \epsilon, 1\ ]$

II) m dominates all of Bob's other strategies

I) $2m-1+\epsilon$ dominates all of Alice's other strategies

This leaves Alice with the strategy: $2m - 1 + \epsilon$,

and Bob with the strategy: m.

So the value of the game is b.

Next we might consider $m \in [2/3, 3/4)$, because at $m = 3/4$, Bob's strategy of betting 1/4 dominates his strategy of betting 0.

2.2.3 $m \in [2/3, 3/4)$

I) $2m - 1 + \epsilon$ dominates $[\ 2m - 1 + \epsilon, 1\ ]$

II) m dominates $(\ 3m - 2 + \epsilon, m\ ]$

I) 0 dominates $[\ 0, 2m - 1\ )$

II) $3m - 2 - \epsilon$ dominates $3m - 2 + \epsilon$ and $3m - 2$; all other remaining strategies are equivalent

I) $2m - 1$ is dominated

This leaves Alice with the strategies: $0, 2m - 1 + \epsilon$,

and Bob with the strategies: $m, 3m - 2 - \epsilon$

This gives us a 2x2 matrix game:

$$\begin{bmatrix} b+c & b \\ 0 & a+b \end{bmatrix}$$

we find their optimal strategies to be

$$Bob's = \left(\frac{a+b}{a+b+c}, \frac{c}{a+b+c} \right)$$

$$Alice's = \left(\frac{a}{a+b+c}, \frac{b+c}{a+b+c} \right)$$

.

2.2.4 In General

In case you have not noticed yet, there is a pattern to the appearance of cases. New strategies crop up around certain points on the graph as m increases. Take $m = 1/2$ for example, the first value for which Bob has a winning strategy. We drop a vertical line down m and it meets $1 - m$ on Bob's axis, which is because $m = 1 - m$ in this case. Then, as m becomes bigger, we can draw a horizontal line across $2m - 1$ (=0 at $m = 1/2$) which $= m - 1$ when $m = 2/3$. Bob acquires another strategy at this point, $3m - 2 = 0$. Then m increases, and we drop a vertical line through $3m - 2$, which $= 1 - m$ at $m = 3/4$.

What is happening is that starting from Alice's $2m - 1$, we draw connected horizontal and vertical lines within the middle region until we reach one of the axes (see figure), and the points these lines intersect on the axes are points around which the optimal strategies lie. A change of axis upon which our series of lines ends on is a change of the case with which we are dealing, though in some cases, such as $m = 1/2$, when the series ends at $1 - m$ it is a separate case unto itself.

Now, how do we find these points around which the optimal strategies lie? Again starting with Alice's $2m - 1$ on the lower boundary line where $b_2 = m$, and moving horizontally to the upper boundary line, what is the b_2 for that point? The equation for the upper boundary line is $b_2 = b_1 + (m - 1)$, and here $b_1 = 2m - 1$, so $b_2 = 3m - 2$. Then, dropping from this point down to the lower boundary line, $b_1 = b_2 + (m - 1)$, we find that $b_1 = 4m - 3$ there. Moving horizontally, the next $b_2 = 5m - 4$, etc... By now the pattern of optimal centrepoints is evident:

$$b_1\,'s = 2km - (2k - 1);$$

$$b_2\,'s = (2k + 1)m - 2k;$$

where k is a positive integer. These points "break" when

$$2km - (2k - 1) = 0 \Rightarrow m = \frac{2k - 1}{2k}$$

and

$$(2k + 1)m - 2k = 0 \Rightarrow m = \frac{2k}{2k + 1};$$

$$\Longrightarrow m = \frac{k}{k + 1},$$

which is when $m = 1/2, 2/3, 3/4, 4/5, 5/6...$

Include General Solution, once I make out what Bradley has written on the page...

This would mean that as $m \to 1$, there are an infinite number of these points. That makes for a lot of optimal strategies. The matrices for larger m become too unwieldy to present here, but they also become unnecessary. Remember that Alice wants the combination of strategies to fall within the middle region. However, as this region shrinks and the number of strategies increases, it becomes easy for Bob to make sure that the payoff does not fall within the middle region. The best Alice can really hope for at that point is to place in the lower of $a + b$ or $b + c$, which she can do. By betting $[0, 1 - m - \epsilon]$ she guarantees $b + c$, and by betting $[2m - 1 + \epsilon]$ she guarantees $a + b$. Bob will have to accept that as his payoff, which is not so bad. His payoff has increased significantly as m increased. At $m = 1/2$, he received $min\{b, (b + c)/2\}$, and now he receives $min\{a + b, b + c\}$.

2.2.5 $m = 1$

This really is simply solving the 2x2 matrix game:

$$\begin{bmatrix} a + b & \frac{a+2b+c}{2} \\ \frac{1}{2} & b + c \end{bmatrix}$$

which we can solve...

The solution we get is:

- Both players bet 1 if $b + c > a + b$.
- Alice bets nothing and Bob bets 1 if $b + c \leq a + b \leq 1/2$.
- Alice bets 1 and Bob bets nothing if $a + b \geq b + c \geq 1/2$.
- Both players bet 0 if $b + c < 1/2 < a + b$.

The reason why it's just a 2x2 is because it makes no sense to bet somewhere in between 0 and 1. If Bob is going to bet, it means he wants the payoff $b+c$ rather than $a+b$, and the only way to guarantee that his payoff will not be $a+b$ is to bet 1. Likewise, the only way to guarantee that his payoff will not be $b+c$ is to bet 0. Therefore, each player has only the strategies 0 and 1.

3 Conclusions

3.1 Summing up

While we have successfully analyzed the two-person game, several questions remain:

1. **One Shot Deal -** Jeopardy is a game which you either win or lose. There is no replay of the game, no best 2-out-of-3; you either come back the next day or go home today. The unfortunate thing about this is that though our optimal strategies will guarantee the best payoff over many games, the fact remains that you will either get a payoff of 1 or 0, not p_2 or $1 - p_1$ or $(1 - p_1)p_2$.

2. **Does Not Maximize Money -** Our solution does not maximize a player's money at the end of the game. We have ignored the greed factor, but this is because we have placed the emphasis on actually winning rather than risking one's winnings.

3. **What about the 3-person game? -** It has been disappointing, in a way, not to have had the time and space to solve the 3-person game here, but it has been exciting, nevertheless, because the 2-person game proved to be so interesting.

   The 3-person game is really just an extension of the 2-person game, and the methods used to solve the 2-person game may be expanded in order to solve the 3-person game. Of course, we are comparing planes rather than lines for domination, and the addition of the third player may cause different "break points" for the cases. Then, after the game is solved, we could see what effect a nice consolation prize worth $0 < c < 1$ for the second place finisher has on the betting strategies, for it might become better to play for the consolation prize than the win.

4. **What about real game situations? -** At the time of the writing of this paper, we do not have data from the actual game show to work with.

Despite the shortcomings in our analysis of betting strategies in Final Jeopardy, it is still useful. Our solutions offer the assurance that you will not lose due to playing a poor strategy, and cannot be second guessed by every home in America.