Abstract

Recently, Garsia and Haglund proved a long-standing conjecture of Garsia and Haiman that a certain symmetric rational function \( C_n(q,t) \) is a polynomial with positive integer coefficients. When \( q = t = 1 \), the polynomial \( C_n(q,t) \) evaluates to the Catalan number, so has come to be known as the \( q,t \)-Catalan polynomial. Garsia and Haiman’s proof was based on a recurrence suggested by a pair of statistics on \( \mathcal{D}_n \), the set of Catalan paths of length \( 2n \). Garsia and Haglund’s proof was based on a recurrence suggested by a pair of statistics on \( \mathcal{D}_n \), the set of Catalan paths of length \( 2n \). Garsia and Haiman had shown that \( C_n(q,1) \) is the generating function for the area statistic area(II), which counts the

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number of lattice squares below a Catalan path and above the line $y = x$. A second statistic $\text{maj}(\beta(\Pi))$ on Catalan paths was introduced by Haglund who, with Garsia, proved that $C_n(q, t) = \sum_{\Pi \in D_n} q^{\text{area}(\Pi)} t^{\text{maj}(\Pi)}$. We generalize Haglund’s statistic to $S_{n,d}$, the set of Schröder paths from $(0,0)$ to $(n,n)$ with diagonal steps.

We develop recurrence relations and prove that on the set $S_{n,d}$, this new statistic is equidistributed with a known extension of the area statistic to Schröder paths. We conjecture that these statistics are also symmetric:

$$\sum_{\Pi \in S_{n,d}} q^{\text{area}(\Pi)} t^{\text{new}(\Pi)} = \sum_{\Pi \in S_{n,d}} q^{\text{area}(\Pi)} t^{\text{new}(\Pi)}.$$ 

In addition we prove that if $t = 1/q$ this sum equals a product of $q$-binomial coefficients.
1 Introduction

The $n$th Catalan number $c_n$ may be defined by the formula $c_n = \frac{1}{n+1} \binom{2n}{n}$. The Catalan numbers arise in many combinatorial settings; indeed, between [?, pp. 172, 212–217, 219] and [?], Stanley provides more than 85 combinatorial interpretations of $c_n$. In this paper we will use just one of these, viewing $c_n$ as the number of lattice paths from $(0,0)$ to $(n,n)$ which consist only of north $(0,1)$ and east $(1,0)$ steps and which do not pass below the line $y = x$. We will call such a path a Catalan path; we write $D_n$ to denote the set of Catalan paths from $(0,0)$ to $(n,n)$.

In [?] Garsia and Haiman introduced a symmetric polynomial $C_n(q,t)$ with nonnegative integer coefficients which satisfies $C_n(1,1) = c_n$. They proved that $C_n(q,1)$ satisfies the recurrence of the Carlitz-Riordan [?] $q$-Catalan polynomial $C_n(q)$,

$$C_n(q) = \sum_{k=1}^{n} q^{k-1} C_{k-1}(q) C_{n-k}(q), \quad C_0(q) = 1,$$

and obtained the formula

$$q\binom{n}{2} C_n(q,1/q) = \frac{1}{[n+1]_q} \left[\begin{array}{c} 2n \\ n \end{array}\right]_q,$$  \hspace{1cm} (2)

where $[n]_q$ is the $q$-binomial coefficient: $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$ for $n \geq 1$; $[0]_q! := 1$ and $[n]_q! := [1]_q [2]_q \cdots [n]_q$ for $n \geq 1$; $\left[\begin{array}{c} n \\ k \end{array}\right]_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}$. In view of (1) and (2), $C_n(q,t)$ is called the $q,t$-Catalan polynomial.

In order to define $C_n(q,t)$, we introduce some terminology and notation, which we illustrate in Figure 1 below. Let $\mu$ denote a partition of $n$ and let $s$ denote a cell in the Ferrers diagram of $\mu$. By the arm (respectively, co-arm) of $s$ we mean the set of cells in the same row as $s$ and strictly to the right (respectively, left) of $s$. By the leg (respectively, co-leg) of $s$ we mean the set of cells in the same column as $s$ and strictly below (respectively, above) $s$. When $s$ has been specified, we let $a$ (respectively, $a'$) denote the number of cells in the arm (respectively, co-arm) of $s$, and $l$ (respectively, $l'$) denote the number of cells in the leg (respectively, co-leg) of $s$. For example, for the cell labeled $s$ in Figure 1 we have $a = 5$, $a' = 4$, $l = 3$ and $l' = 2$.

Garsia and Haiman defined $C_n(q,t)$ by

$$C_n(q,t) = \sum_{\mu \vdash n} t^2 \frac{q^a (1-t)(1-q) \prod_{i=0}^{l-1} (1-q^{a'+l'}) (\sum q^{a'+l'})}{\prod (q^a - t^{a+1}) (t^l - q^{a+1})}.$$  \hspace{1cm} (3)

Here the symbol $\Pi(0,0)$ represents the product over all cells in the Ferrers diagram of $\mu$ except the upper left corner. All other sums and products within the $\mu$th summand are over all cells in the Ferrers diagram of $\mu$. In Example 1 below we use (3) to compute $C_3(q,t)$. 

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responds to the sign character $\chi$ and consider the terms in (3) indexed by the conjugate Schur function indexed by $\lambda$. Since conjugation interchanges $a$ with $l$ and $a'$ with $l'$, the term indexed by $\mu$ in $C_n(q, t)$ is equal to the term indexed by $\mu'$ with $q$ and $t$ interchanged. Thus, $C_n(q, t) = C_n(t, q)$. This is exemplified in Figure 2, with the cell corresponding to $s$ labeled $s'$. Note here that $l = 5$, $l' = 4$, $a = 3$ and $a' = 2$.

It is clear from (3) that $C_n(q, t)$ is a rational function, but in fact much more is true: $C_n(q, t)$ is a polynomial with nonnegative integer coefficients. Haiman was the first to prove that $C_n(q, t)$ is a polynomial [1]; his techniques come from algebraic geometry. Garsia and Haglund were the first to show that the coefficients in $C_n(q, t)$ are nonnegative integers [2], which they did by showing that $C_n(q, t)$ is the generating function for the set of Catalan paths with respect to two statistics, $area$ and $dmaj$. (We describe these statistics in Section 2.) A short summary of this proof appeared in [3].

Another method of proving that $C_n(q, t)$ has nonnegative integer coefficients involves a more general formula for the Frobenius series $\mathcal{F}_n(q, t)$ of the module of diagonal harmonics defined by Haiman in [2]. Haiman [2] recently proved that

$$\mathcal{F}_n(q, t) = \sum_{\mu \vdash n} \tilde{H}\mu[X; q, t] \frac{t^q a(1 - t)(1 - q) \prod_{i=0}^{n-1}(1 - q^{a_i}t^i)}{\prod_i (q^a - t)(1 - q^{a_i})} \left( \sum q^{a_i}t^i \right).$$

(4)

Here $\tilde{H}\mu[X; q, t] = \sum_\lambda \tilde{K}\lambda,\mu(q, t)s_\lambda$ is the modified Macdonald polynomial; $s_\lambda[X]$ is the Schur function indexed by $\lambda$; $\tilde{K}\lambda,\mu(q, t) = t^{-1} K\lambda,\mu(q, 1/t)$; and $K\lambda,\mu(q, t)$ is the $q, t$-Kostka number, defined by Macdonald [7].

Equation (4) was initially conjectured by Garsia and Haiman [2]. They defined a linear operator $\nabla$ on the basis $\{\tilde{H}\mu[X; q, t]\}_{\mu}$ of the space of diagonal harmonics by $\nabla \tilde{H}\mu[X; q, t] = t^{-1} q^a \tilde{H}\mu[X; q, t]$, and showed that the right hand side of (4) is equal to $\nabla e_n[X]$, where $e_n$ is the $n$th elementary symmetric function. Since the coefficient of $s_1$ in $\tilde{K}\lambda,\mu(q, t) = t^{-1} q^a e_n$, taking the coefficient of $s_1$ on the right-hand side of (4) yields the rational function $C_n(q, t)$. It follows that the component of $\mathcal{F}_n(q, t)$ which corresponds to the sign character $\chi^{1^n}$ is equal to $C_n(q, t)$. Therefore $C_n(q, t)$ has nonnegative coefficients.
Regarding Garsia and Haglund's proof, since $C_n(q, 1)$ satisfies (1), it is not difficult to show that

$$C_n(q, 1) = \sum_{\Pi \in \mathcal{D}_n} q^{\text{area}(\Pi)},$$

where the sum is over the set $\mathcal{D}_n$ of all Catalan paths from $(0, 0)$ to $(n, n)$. This observation led to a search for a second statistic $b$ on Catalan paths for which

$$C_n(q, t) = \sum_{\Pi \in \mathcal{D}_n} q^{\text{area}(\Pi)} t^{b(\Pi)}.$$

In [?] Haglund introduced the statistic $d\text{maj}$ and conjectured that

$$F_n(q, t) := \sum_{\Pi \in \mathcal{D}_n} q^{\text{area}(\Pi)} t^{d\text{maj}(\Pi)}$$

is equal to $C_n(q, t)$. Letting $F_{n,p}(q, t)$ denote the restriction of $F_n(q, t)$ to those paths in $\mathcal{D}_n$ which end with a north step followed by $p$ east steps, Haglund showed that

$$F_{n,p}(q, t) = \sum_{r=0}^{n-p} \left[ \begin{array}{c} r + p - 1 \\ r \end{array} \right] q^{(\frac{r}{2})} t^{n-p} F_{n-p,r}(q, t). \quad (5)$$

Garsia and Haglund then showed that the coefficients of $s_{1^{n-p}}$ in

$$Q_{n,p}(q, t) = t^{n-p} q^{\left(\frac{r}{2}\right)} \nabla e_{n-p} \left( \frac{X^{1-q^p}}{1-q} \right)$$

also satisfy (5). A simple corollary is that $F_n(q, t) = C_n(q, t)$.

Interpreting the symmetry of $C_n(q, t)$ in terms of the statistics $\text{area}$ and $d\text{maj}$, we find that

$$|\{\Pi \in \mathcal{D}_n \mid \text{area}(\Pi) = r, d\text{maj}(\Pi) = s\}| = |\{\Pi \in \mathcal{D}_n \mid \text{area}(\Pi) = s, d\text{maj}(\Pi) = r\}|$$

for all integers $r$ and $s$. However, a constructive involution $\omega : \mathcal{D}_n \rightarrow \mathcal{D}_n$ such that $\text{area}(\Pi) = d\text{maj}(\omega(\Pi))$ for all $\Pi \in \mathcal{D}_n$ has not yet been discovered. In [?], Haglund gave a combinatorial proof that $\text{area}$ and $d\text{maj}$ are equidistributed on $\mathcal{D}_n$, thus providing a combinatorial proof that $C_n(q, 1) = C_n(1, q)$.

In this paper we provide a natural extension of the statistics $\text{area}$ and $d\text{maj}$ to the class of lattice paths known as Schröder paths: paths from $(0, 0)$ to $(n, n)$ which consist only of north $(0, 1)$, east $(1, 0)$, and diagonal $(1, 1)$ steps and which do not pass below the line $y = x$. For example, one Schröder path from $(0, 0)$ to $(10, 10)$ is the following.
For fixed \( n \), the number of Schröder paths is known to be equal to the large Schröder number \( r_n \). The generating function for \( r_n \) is given by

\[
\sum_{n \geq 0} r_n x^n = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}
\]

and the initial sequence is \((r_0, r_1, r_2, \ldots) = (1, 2, 6, 22, 90, \ldots)\). The Schröder numbers are related to the Catalan numbers in that

\[
r_n = \sum_{d=0}^{n} \binom{2n-d}{d} c_{n-d} \quad (n \geq 0).
\]

To illustrate our general approach, observe that we can prove (6) by counting Schröder paths according to how many diagonal steps they contain. Specifically, to prove (6) it is sufficient to show that for \( 0 \leq d \leq n \), the number of Schröder paths from \((0,0)\) to \((n,n)\) which contain exactly \( d \) diagonal steps is \( \binom{2n-d}{d} c_{n-d} \). To form a Schröder path with \( d \) diagonal steps, first choose a path in \( \mathbb{D}_{n-d} \), which can be done in \( c_{n-d} \) ways, and then insert the \( d \) diagonal steps into the chosen path, which can be done in \( \binom{2n-d}{d} \) ways.

The area statistic on \( \mathbb{D}_n \) has a natural extension to the set of Schröder paths. In Section 2 we describe this extended area statistic in detail and define a new statistic called smaj on the set \( S_{n,d} \) of Schröder paths from \((0,0)\) to \((n,n)\) which contain exactly \( d \) diagonal steps. Expanding on Haglund’s proof that area and \( dmaj \) are equidistributed on Catalan paths, we show in Section 3 that area and smaj are equidistributed on \( S_{n,d} \) and we conjecture that the generating function

\[
S_{n,d}(q,t) = \sum_{\Pi \in S_{n,d}} q^{\text{area}(\Pi)} t^{\text{smaj}(\Pi)}
\]

is symmetric in \( q \) and \( t \). We then restrict \( S_{n,d}(q,t) \) to those paths which end with exactly \( p \) east steps; in Theorem 7 we show that this restriction satisfies a recurrence which is similar to Haglund’s recurrence (5).

In [?], Bonin, Shapiro, and Simion proved that the number of paths \( \Pi \in S_{n,d} \) is

\[
S_{n,d}(1,1) = \frac{1}{n} \binom{n}{d} \binom{2n-d}{n-1}.
\]
They also considered the major index \( \text{maj}(\Pi) \) of a Schröder path \( \Pi \in \mathcal{S}_{n,d} \). To define \( \text{maj}(\Pi) \), first fix a total ordering on the alphabet \( \{E, D, N\} \). If we view the path \( \Pi \) as a word in this alphabet then \( \text{maj}(\Pi) \) is the sum of the positions in which \( \Pi \) has a descent. For example, taking \( E < D < N \), we observe that \( \Pi = NDENNEDN \) has descent set \( \{1, 2, 6\} \), so \( \text{maj}(\Pi) = 9 \). Bonin, Shapiro, and Simion showed that with respect to the ordering \( E < D < N \), the distribution of \( \text{maj} \) on Schröder paths is given by

\[
\sum_{\Pi \in \mathcal{S}_{n,d}} q^{\text{maj}(\Pi)} = \frac{1}{[n-d+1]q} \left[ \frac{2(n-d)}{n-d} \right]_q \left[ \frac{2n-d}{d} \right]_q.
\]

Specializing this result to Catalan paths, they also showed that

\[
\sum_{\Pi \in \mathcal{D}_n} q^{\text{maj}(\Pi)} = \frac{1}{[n+1]q} \left[ \frac{2n}{n} \right]_q.
\]

Motivated by these results and Garsia and Haiman’s observation (2), we prove in Theorem 3 that

\[
q^{(n+d-1)(n-d)/2} \mathcal{S}_{n,d}(q, 1/q) = \frac{1}{[n-d+1]q} \left[ \frac{2(n-d)}{n-d} \right]_q \left[ \frac{2n-d}{d} \right]_q.
\]

A third statistic on \( \mathcal{D}_n \), called \( \text{dinv} \), was defined on Catalan paths by Haiman, who conjectured that \( C_n(q, t) = \sum_{\Pi \in \mathcal{D}_n} q^{\text{area}(\Pi)} t^{\text{dinv}(\Pi)} \). Garsia, Haiman, and Haglund showed that while \( \text{dinv} \) and \( \text{dmaj} \) are not necessarily equal, they are equidistributed on \( \mathcal{D}_n \). Haiman’s conjecture follows from Garsia and Haglund’s proof that \( C_n(q, t) = \sum_{\Pi \in \mathcal{D}_n} q^{\text{area}(\Pi)} t^{\text{dinv}(\Pi)} \).

Recently Haglund and Loehr [?] extended the statistics \( \text{area} \) and \( \text{dinv} \) to the set \( \mathcal{P}_n \) of parking functions on \( n \) cars, as defined in [?, p. 94], and conjectured that the Hilbert series \( \mathcal{H}_n(q, t) \) of the space of diagonal harmonics is described combinatorially by the formula \( \mathcal{H}_n(q, t) = \sum_{P \in \mathcal{P}_n} q^{\text{area}(P)} t^{\text{dinv}(P)} \). They also phrased their conjecture in terms of the statistic \( \text{dmaj} \) and proved that the two forms of their conjecture are equivalent. As a special case, they gave a combinatorial proof that

\[
\sum_{\Pi \in \mathcal{D}_n} q^{\text{area}(\Pi)} t^{\text{dinv}(\Pi)} = \sum_{\Pi \in \mathcal{D}_n} q^{\text{dmaj}(\Pi)} t^{\text{area}(\Pi)}.
\]

Specifically, they defined a bijection \( \phi : \mathcal{D}_n \rightarrow \mathcal{D}_n \) such that \( \text{area}(\Pi) = \text{dmaj}(\phi(\Pi)) \) and \( \text{dinv}(\Pi) = \text{area}(\phi(\Pi)) \). In Section 4 we define a statistic \( \text{sinv} \) on Schröder paths which is equal to \( \text{dinv} + n \) when restricted to Catalan paths. We then extend \( \phi \) to \( \mathcal{S}_{n,d} \) in such a way that \( \text{area}(\Pi) = \text{sma}(\phi(\Pi)) \) and \( \text{sinv}(\Pi) = \text{area}(\phi(\Pi)) \).

In Section 5 we mention a few conjectures and open problems related to the results in this paper.
2 Defining the Statistics

Throughout this paper we will use the following notation. We write \( D_n \) to denote the set of lattice paths from \((0, 0)\) to \((n, n)\) which consist only of north \((0, 1)\) and east \((1, 0)\) steps and which do not pass below the line \( y = x \). We refer to such paths as Catalan paths. We write \( S_{n,d} \) to denote the set of lattice paths from \((0, 0)\) to \((n, n)\) which consist only of north \((0, 1)\), east \((1, 0)\), and diagonal \((1, 1)\) steps, which do not pass below the line \( y = x \), and which include exactly \( d \) diagonal steps. We refer to such paths as Schröder paths.

Our goal is to define statistics \( a \) and \( b \) on \( S_{n,d} \) such that the distribution function 
\[
S_{n,d}(q, t) = \sum_{\Pi \in S_{n,d}} q^{a(\Pi)} t^{b(\Pi)}
\]
is symmetric in \( q \) and \( t \). In addition, since \( S_{n,0} = D_n \), we would like \( S_{n,0}(q, t) \) to be equal to \( C_n(q, t) \), up to constant powers of \( q \) and \( t \). In this section we introduce candidates for \( a \) and \( b \). We begin by recalling the area statistic on \( D_n \).

**Definition 1** Given a Catalan path \( \Pi \in D_n \), we write \( \text{area}(\Pi) \) to denote the number of unit squares that lie below \( \Pi \) and above the line \( y = x \).

In Figure 1 the squares counted by the area statistic are shaded; the illustrated path has area 13.

![Figure 1: A Catalan path with an area statistic of 13](image)

In \cite{GarsiaHaiman1994} Garsia and Haiman showed that when \( t = 1 \) the \( q, t \)-Catalan polynomial reduces to the generating function
\[
C_n(q, 1) = \sum_{\Pi \in D_n} q^{\text{area}(\Pi)}.
\]
This observation led to a search for a second statistic \( b \) on \( D_n \) for which
\[
C_n(q, t) = \sum_{\Pi \in D_n} q^{\text{area}(\Pi)} t^{b(\Pi)}.
\]
Haglund \cite{Haglund2002} eventually found such a statistic, which he called \( \text{dmaj} \). Next we recall \( \text{dmaj} \), renormalized so that \( \text{dmaj}(\Pi) = b(\Pi) + n \) when \( \Pi \in D_n \).
Definition 2  A Catalan path in $D_n$ is said to be balanced whenever it has the form

$$N^{b_1} E^{b_1} N^{b_2} E^{b_2} \cdots N^{b_m} E^{b_m},$$

where $N$ and $E$ represent north and east steps, respectively, and $b_1 + b_2 + \cdots + b_m = n$. For each Catalan path $\Pi$, we write $\beta(\Pi)$ to denote the balanced path constructed by tracing the path of a billiard ball shot straight west from the point $(n, n)$, following the rules:

1. A billiard ball headed west is reflected straight south when it encounters the top of a southbound step.
2. A billiard ball headed south is reflected straight west when it hits the line $y = x$.

We define $d\text{maj}(\Pi) = b_1 + (b_1 + b_2) + (b_1 + b_2 + b_3) + \cdots + (b_1 + b_2 + \cdots + b_{n-1}) + n$.

In practice one computes $d\text{maj}(\Pi)$ by summing the $x$ coordinates of the corners of $\beta(\Pi)$ which lie on the line $y = x$. We will sometimes view $\beta(\Pi)$ as a partition $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$ of $d\text{maj}(\Pi)$ into distinct parts.

Figure 2 above shows a Catalan path $\Pi$ with its associated balanced path $\beta(\Pi)$, given by the dotted line. This Catalan path has $d\text{maj}(\Pi) = 10 + 8 + 4 + 3 + 1 = 26$ and $\beta(\Pi) = (10, 8, 4, 3, 1)$.

![Figure 2: A Catalan path $\Pi$ and the corresponding path $\beta(\Pi)$ with $d\text{maj}(\Pi) = 26$.](image)

The relation

$$r_n = \sum_{d=0}^{n} \binom{2n - d}{d} c_{n-d}$$

leaves us to think of the set of Schröder paths from $(0, 0)$ to $(n, n)$ as being built from Catalan paths of length $n - d$ by inserting $d$ diagonal steps. Our approach to extending the area and $d\text{maj}$ statistics from Catalan paths to Schröder paths is similar: we will build the statistics on $S_{n,d}$ by starting with the area and $d\text{maj}$ statistics on $S_{n-d,0} = D_{n-d}$ and then determining the effect of inserting $d$ diagonal steps. We begin with area.
Definition 3 For all $\Pi \in S_{n,d}$ we define $\text{area}(\Pi)$ to be the number of upper triangles between $\Pi$ and the line $y = x$. That is, $\text{area}(\Pi)$ is the number of triangles with vertices $(i, j)$, $(i, j + 1)$ and $(i + 1, j + 1)$ such that $i \leq j$ and $(i, j + 1)$ is weakly below $\Pi$. It will be convenient to define a partition of $\text{area}(\Pi)$, $\alpha(\Pi) = <1^{a_1}2^{a_2} \cdots k^{a_k}>$, in which $a_i$ is the number of rows which contain exactly $i$ triangles contributing to $\text{area}(\Pi)$.

In Figure ?? below, $\alpha(\Pi) = <1^32^33^14^1>$ and $\text{area}(\Pi) = 38$.

We remark that when we view a Catalan path $\Pi$ as an element of $S_{n,0}$, we obtain slightly different statistics than those of Garsia and Haglund’s paper. In particular, we have added the $n$ upper triangles whose hypotenuses lie on the line $y = x$ to $\text{area}(\Pi)$. In order to correct for this, we have also added $n$ to Haglund’s statistic $\text{dmaj}$. This amounts to summing the labels on all of the diagonal points in $\beta(\Pi)$, including the endpoints.

Given a Schröder path $\Pi$, let $\Pi'$ denote the Catalan path obtained by deleting the diagonal steps in $\Pi$, let $\alpha(\Pi')$ denote the area partition of $\Pi'$, and let $\beta(\Pi')$ denote the balanced path associated with $\Pi'$. Figure 3 below shows $\Pi'$ for the path $\Pi$ given in Figure ??.

For this path we have $\alpha(\Pi') = <1^22^34^1>$ and $\text{area}(\Pi') = \sum_{i=1}^{n} ia_i = 22$. In addition, $\beta(\Pi') = N^2E^2N^1E^1N^3E^3N^3E^3$, or $\beta = (9, 6, 3, 2)$, and $\text{dmaj}(\Pi') = 2 + 3 + 6 + 9 = 20$.

Figure 3: The path $\Pi'$ with $\text{area} = 22$ and $\text{dmaj} = 20$

Haglund’s proof that $\text{area}$ and $\text{dmaj}$ are equidistributed on $D_n$ amounts to showing that the number of Catalan paths associated with the balanced path $\beta = (\beta_1, \ldots, \beta_k)$ is equal to the number of Catalan paths whose area partition $\alpha(\Pi)$ is the conjugate of $\beta$, $\beta' = <1^{\beta_1-\beta_2}2^{\beta_2-\beta_3} \cdots k^{\beta_k}>$. We build on this proof by keeping track of the effect on $\text{area}$ and $\text{dmaj}$ of adding diagonal steps. To do this, we introduce the height vector of a Schröder path.

Definition 4 For any lattice point $(x, y)$, we write $\text{height}(x, y) = y - x$. The height vector of a Schröder path $\Pi$ is given by $h(\Pi) = (h_0, h_1, \ldots, h_k)$, where $h_i$ is the number of diagonal steps in $\Pi$ which have been inserted into $\Pi'$ at height $i$. 
The effect on area of inserting a diagonal step into $\Pi'$ at height $i$ is to add $i$ to the area statistic, so that $area(\Pi) = area(\Pi') + \sum_{i=0}^{k} ih_i$. In Figure 4 below we have $h(\Pi) = (2, 1, 3, 3, 0)$ and the additional units of area added by the insertion of diagonal steps into $\Pi'$ are shaded. To extend $dmaj(\Pi)$ to Schröder paths, we mirror this effect by counting the number of peaks in the balanced path which are shifted up and to the right when a diagonal step is inserted.

**Definition 5** To each diagonal step in $\Pi \in S_{n,d}$ which begins at $(x, y)$ in $\Pi'$ we assign a nonnegative integer shift $(x, y)$ which counts the number of $N\rightarrow E$ corners in $\beta(\Pi')$ which are strongly above and weakly to the right of $(x, y)$. The shift vector of $\Pi$ is given by $s(\Pi) = (s_0, s_1, \ldots, s_k)$, where $s_i$ is the number of diagonal steps in $\Pi$ with shift $(x, y) = i$. For any Schröder path $\Pi$ we write

$$sma_i(\Pi) := dmaj(\Pi') + \sum_{i=0}^{k} is_i.$$ 

![Figure 4: A path $\Pi$ with the corresponding Catalan path $\Pi'$](image)

For the path $\Pi$ in Figure 4 we have $s(\Pi) = (1, 3, 2, 2, 1)$. The table below gives both the height and the shift of each of the diagonal steps labeled in Figure 4.
Diagonal (height, shift)
\[\begin{align*}
D_1 & = (0, 4) \\
D_2 & = (2, 3) \\
D_3 & = (0, 3) \\
D_4 & = (1, 2) \\
D_5 & = (2, 2) \\
D_6 & = (3, 1) \\
D_7 & = (3, 1) \\
D_8 & = (3, 1) \\
D_9 & = (2, 0)
\end{align*}\]

3 Properties of the Statistics \textit{area} and \textit{smaj}

In this section we study the generating function \(S_{n,d}(q,t)\), which is given by
\[
S_{n,d}(q,t) = \sum_{\Pi \in \mathcal{S}_{n,d}} q^{\text{area}(\Pi)} t^{\text{smaj}(\Pi)}.
\]

In Theorem 1 we give a combinatorial proof that \(S_{n,d}(q,1) = S_{n,d}(1,q)\). In Theorem 2 we give a recurrence for \(S_{n,d}(q,t)\) in terms of paths which end in exactly \(p\) east steps. In Theorem 3 we evaluate \(S_n(q,1/q)\). In addition to these results, we also make the following conjecture, which has been verified (using Maple) for \(n \leq 12\).

\textbf{Conjecture 1} For all \(n \in \mathbb{Z}_{>0}\) and all \(d \in \mathbb{Z}_{\geq 0}\) such that \(n \geq d\) we have \(S_{n,d}(q,t) = S_{n,d}(t,q)\).

A constructive proof of Conjecture 1 would include (when \(d = 0\)) a constructive proof that \textit{area} and \textit{dmaj} are symmetrically distributed on Catalan paths. In support of our conjecture, we offer Theorem 1. In the proof of Theorem 1 we first enumerate the Schröder paths which have a given shift vector and whose corresponding Catalan path has a given balanced path. We then enumerate the Schröder paths whose corresponding Catalan path has a given area partition and height vector. When the shift and height vectors are equal and the balanced path and area partition are conjugates, these sets are equinumerous. Taking the appropriate sums, we find that \textit{area} and \textit{smaj} are equidistributed on \(\mathcal{S}_{n,d}\).

\textbf{Lemma 1} For each \(\Pi \in \mathcal{S}_{n,d}\), let \(\Pi' \in \mathcal{D}_{n-d}\) denote the Catalan path obtained from \(\Pi\) by removing the diagonal steps.

(a) Fix a balanced path \(\beta = (\beta_1, \ldots, \beta_k)\) in \(\mathcal{D}_{n-d}\) and a shift vector \(s = (s_1, \ldots, s_k)\). Let \(P_{\beta,s} = \{\Pi \in \mathcal{S}_{n,d} | \beta(\Pi') = \beta \text{ and } s(\Pi) = s\}\). Taking \(\beta_0 = n\) and \(\beta_{k+1} = \beta_{k+2} = 0\), we have
\[
|P_{\beta,s}| = \prod_{i=1}^{k-1} \frac{(\beta_i - \beta_{i+2} - 1)}{(\beta_{i+1} - \beta_{i+2})} \prod_{i=0}^{k} \frac{(\beta_i - \beta_{i+2} + s_i - 1)}{s_i}.
\]
(b) Fix an area partition $\alpha = < 1^{a_1} 2^{a_2} \cdots k^{a_k} >$ and a height vector $h = (h_0, h_1, \ldots, h_k)$. Let $\tilde{P}_{\alpha,h} = \{ \Pi \in S_{n,d} | \alpha(\Pi) = \alpha \text{ and } h(\Pi) = h \}$. Taking $a_0 = 0$ and $a_k+1 = 0$, we have

$$|\tilde{P}_{\alpha,h}| = \prod_{i=1}^{k-1} \left( \frac{a_i + a_{i+1} - 1}{a_i + 1} \right) \prod_{i=0}^{k} \left( \frac{a_i + a_{i+1} + h_i - 1}{h_i} \right).$$

Furthermore, if $\alpha = \beta' = < 1^{\beta_1} 2^{\beta_2} \cdot k^{\beta_k} - 0 >$ and $h = s$ then $|\tilde{P}_{\alpha,h}| = |P_{\beta,s}|$.

**Proof.**

(a) For each $i$ ($1 \leq i \leq k - 1$) there is a rectangle above $\beta$ of size $(\beta_i - \beta_{i+1} - 1) \times (\beta_{i+1} - \beta_{i+2})$ within which any Catalan path $\Pi'$ having balanced path $\beta$ chooses $\beta_{i+1} - \beta_{i+2}$ east steps (see the diagram below). The path $\Pi'$ is uniquely determined by the choices of these steps in each of the $k - 1$ rectangles. Thus, the number of Catalan paths in $D_n$ whose associated balanced path is $\beta$ is equal to $\prod_{i=1}^{k-1} \left( \beta_i - \beta_{i+2} - 1 \right)$.

Given a Catalan path $\Pi'$ with $\beta(\Pi') = \beta$, we wish to count the number of ways to insert $d$ diagonal steps so that the shift vector of the resulting Schröder path is $s$. Within each $(\beta_i - \beta_{i+1} - 1) \times (\beta_{i+1} - \beta_{i+2})$ rectangle in $\beta(\Pi')$, there are $(\beta_i - \beta_{i+1} - 1) + (\beta_{i+1} - \beta_{i+2}) - 1$ north and east steps among which $s_i$ diagonal steps having shift $i$ must be inserted. The number of ways to insert diagonal steps into $\Pi'$ so that the shift vector is $s$ is therefore given by the second product $\prod_{i=0}^{k} \left( \frac{\beta_i - \beta_{i+2} + s_i - 1}{s_i} \right)$.

(b) The number of Catalan paths with area partition $\alpha$ is the number of multiset permutations $\gamma = \gamma_1 \gamma_2 \cdots \gamma_m$ of $\{1^{a_1}, 2^{a_2}, \ldots, k^{a_k}\}$ which end with a 1 and have no descents of size two or more. That is, $\gamma_m = 1$ and $\gamma_i - \gamma_{i+1} \leq 1$. This is the number of ways to rearrange the rows of $\alpha$ and still get a Catalan path. Given a permutation of $\{1^{a_1}, \ldots, i^{a_i}\}$ which ends with a 1 and has no descents of size two or more, each $i+1$ must be inserted immediately to the left of an $i$. There are $\binom{a_i + a_{i+1} - 1}{a_{i+1}}$ ways to do this, so $\prod_{i=1}^{k-1} \left( \frac{a_i + a_{i+1} - 1}{a_{i+1}} \right)$ Catalan paths in $D_{n-d}$ have area partition $\alpha$. 

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A Catalan path $\Pi'$ with area partition $\alpha$ has $a_i + a_{i+1}$ points at height $i$, so the number of ways to insert $h_i$ diagonal steps into $\Pi'$ at height $i$ is \((a_i + a_{i+1} + h_i - 1)\).

It follows that the number of ways to insert diagonal steps so that the height vector is $h$ is $\prod_{i=0}^{k} \left(a_i + a_{i+1} + h_i - 1\right)$.

If $\alpha = \beta'$ then $a_i + a_{i+1} = \beta_i - \beta_{i+2}$, so $|\tilde{P}_{\beta,s}| = |P_{\beta,s}|$. \hfill \Box

**Theorem 1** For all $n, c \in \mathbb{Z}^\geq 0$ and all $d \in \mathbb{Z}^\geq 0$ such that $n \geq d$ and $c \leq \left(\frac{n+1}{2}\right)$, the number of paths $\Pi \in S_{n,d}$ which have $\text{smaj}(\Pi) = c$ is equal to the number of paths $\Pi \in S_{n,d}$ which have $\text{area}(\Pi) = c$. Equivalently, $S_{n,d}(q,1) = S_{n,d}(1,q)$.

**Proof.** From Lemma 1(a),

\[
S_{n,d}(1,q) = \sum_{\Pi \in S_{n,d}} q^{\text{smaj}(\Pi)} = \sum_{(\beta,s) \in P_{\beta,s}} \sum_{\Pi \in P_{\beta,s}} q^{\text{smaj}(\Pi)}
\]

\[
= \sum_{(\beta,s)} \prod_{i=1}^{k} \left(\beta_i - \beta_{i+2} - 1\right) \prod_{i=0}^{k} \left(\beta_i - \beta_{i+2} + s_i - 1\right) q
\]

Here the outer sum is over all pairs $(\beta,s)$ such that $\beta = (\beta_1,\beta_2,\ldots,\beta_k)$ is a partition with distinct parts whose largest part is $n - d = \beta_1$ and $s = (s_0,s_1,\ldots,s_k)$ is a vector of nonnegative integers such that $\sum_{i=0}^{k} s_i = d$. We remark that there are $2^{n-d-1}$ such partitions $\beta$ and for each $\beta$ of length $k$ there are \(\binom{d+k}{k}\) vectors $s$.

From Lemma 1(b),

\[
S_{n,d}(q,1) = \sum_{\Pi \in S_{n,d}} q^{\text{area}(\Pi)} = \sum_{(\alpha,h) \in P_{\alpha,h}} \sum_{\Pi \in P_{\alpha,h}} q^{\text{area}(\Pi)}
\]

\[
= \sum_{(\alpha,h)} \prod_{i=1}^{k} \left(a_i + a_{i+1} - 1\right) \prod_{i=0}^{k} \left(a_i + a_{i+1} + h_i - 1\right) q
\]

Here the outer sum is over all pairs $(\alpha,h)$ where $\alpha = <1^{a_1}2^{a_2}\ldots k^{a_k}>$ is a partition with $n - d$ parts whose largest part is $k$ such that $a_i > 0$ for $i = 1 \ldots k$ and $h = (h_0,h_1,\ldots,h_k)$ is a vector of nonnegative integers such that $\sum_{i=0}^{k} h_i = d$.

Conjugation $(\alpha,h) \leftrightarrow (\alpha',h)$ provides a bijection between the set of pairs $(\alpha,h)$ and the set of pairs $(\beta,s)$. The conjugate of a partition with $k$ distinct parts whose largest part is $n$ is a partition having $n$ parts and at least one part of each size $i$ for $i = 1,\ldots,k$, and conversely.

In [?] Haglund considered the function $F_{n,p}(q,t) = \sum_{\Pi \in D_{n,p}} q^{\text{area}(\Pi)} t^{\text{smaj}(\Pi)}$, where the sum is over those elements of $D_n$ which end with exactly $p$ east steps, and proved that

\[
F_{n,p}(q,t) = \sum_{r=0}^{n-p} \binom{r+p-1}{r} q^{(p)} t^{n-p} F_{n-p,r}(q,t).
\]
Motivated by this result, we now consider the subset $S_{n,d,p} \subset S_{n,d}$ of Schröder paths which end in a north step followed by exactly $p$ east steps. In Theorem 2, we obtain a recurrence relation for $S_{n,d,p}(q,t) := \sum_{\Pi \in S_{n,d,p}} q^{\text{area}(\Pi)} t^{\text{smaj}(\Pi)}$. Summing over the number of diagonal steps taken after the last north step and the size of the final horizontal step in $\Pi$, we then express $S_{n,d}(q,t)$ in terms of $S_{n-k,d-k,p}(q,t)$.

**Theorem 2** Let $S_{n,d,p}$ denote the set of Schröder paths from $(0,0)$ to $(n,n)$ having exactly $d$ diagonal steps and ending in a north step followed by $p$ east steps. Define

$$S_{n,d,p}(q,t) := \sum_{\Pi \in S_{n,d,p}} q^{\text{area}(\Pi)} t^{\text{smaj}(\Pi)}.$$ 

For all $n, p \in \mathbb{Z}^>0$ and all $d \in \mathbb{Z} \geq 0$ such that $n \geq d + p$,

$$S_{n,d,p}(q,t) = \sum_{r=0}^{n-d} \sum_{m=0}^{d} \binom{r + p + m - 1}{r, p - 1, m} q^{\frac{r+1}{2}} t^n S_{n-p-m,d-m,r}(q,t), \quad (8)$$

$$S_{n,d}(q,t) = \sum_{k=0}^{n-d} \sum_{p=1}^{d} \binom{p + k}{p, k} q S_{n-k,d-k,p}(q,t). \quad (9)$$

**Proof.** To obtain (8), first observe that a path $\Pi$ of the form $\Pi = \Pi_N^p E^p$, where $\Pi \in S_{n-p-m,d-m,r}$, will have $\text{area}(\Pi) = \binom{p+1}{2} + \text{area}(\Pi)$ and $\text{maj}(\Pi) = n + \text{maj}(\Pi)$. A path in the $(p-1) \times r$ rectangular region above the final horizontal step of size $r$ in $\Pi$ will take $r$ east steps and $p-1$ north steps to reach the last north step of $\Pi$. If this path also takes $m$ diagonal steps, then $\binom{r+p+m-1}{r, p-1, m}$ triangles will be added to the area statistic and $m$ shifts will be added to the statistic $\text{maj}$. Summing over the number of diagonal steps inserted and the size of the last horizontal step in $\Pi$, we obtain (8).

To obtain (9), first note that the number of diagonal steps inserted after the last north step may range from 0 to $d$. Similarly, the number of east steps after the last north step may range from 1 to $n - d$. The generating function with respect to $\text{area}$ and $\text{maj}$ for those paths with $p$ east steps and $k$ diagonal steps after the last north step is given by $\binom{p+k}{p, k} q S_{n-k,d-k,p}(q,t)$.
In [?], Garsia and Haiman evaluated $C_n(q, t)$ at $t = 1/q$ to obtain the formula

$$q(\binom{n}{2})C_n(q, 1/q) = \left[\frac{1}{n+1}\right]_{q} \left[2n\right]_{q}. $$

Bonin, Shapiro, and Simion [?] showed that for fixed $n, d \in \mathbb{Z}_{>0}$, the number of paths $\Pi \in S_{n,d}$ is

$$S_{n,d}(1, 1) = \frac{1}{n} \binom{n}{d} \left(\frac{2n-d}{n-1}\right),$$

and gave the distribution of the maj statistic (defined in the introduction) over Schröder paths:

$$\sum_{\Pi \in S_{n,d}} q^{maj(\Pi)} = \left[\frac{1}{n-d+1}\right]_{q} \left[\frac{2(n-d)}{n-d}\right]_{q} \left[\frac{2n-d}{d}\right]_{q}.$$

Motivated by these results, we now show that

$$q^{(n+d-1)(n-d)/2}S_{n,d}(q, 1/q) = \left[\frac{1}{n-d+1}\right]_{q} \left[\frac{2(n-d)}{n-d}\right]_{q} \left[\frac{2n-d}{d}\right]_{q}. $$

We begin by evaluating $S_{n,d,p}(1, 1)$.

**Proposition 1** For all $n, p \in \mathbb{Z}_{>0}$ and all $d \in \mathbb{Z}_{\geq 0}$ such that $n \geq d + p$,

$$S_{n,d,p}(1, 1) = \frac{p}{n} \binom{n}{d} \left(\frac{2n-d-p-1}{n-d-p}\right) = \frac{p(2n-d-p-1)!}{(n-d-p)!(n-d)!d!}. $$

**Proof.** Observe that $S_{n,d,p}(1, 1)$ is equal to the number of Schröder paths from $(0, 0)$ to $(n, n)$ which contain exactly $d$ diagonal steps and whose final $p + 1$ steps are NE$^p$. Let $a$ denote the number of lattice paths from $(0, 0)$ to $(n - d - p, n - d - 1)$ using only north and east steps which never pass below the line $y = x$ and let $b$ denote the number of ways of inserting exactly $d$ diagonal steps into such a path. Then $S_{n,d,p}(1, 1) = ab$. Using a routine reflection argument we find that

$$a = \left(\binom{2n-2d-p-1}{n-p-d}\right) - \left(\binom{2n-2d-p-1}{n-p-d-1}\right),$$

so that

$$a = \frac{p(2n-2d-p-1)!}{(n-p-d)!(n-d)!d!}.$$

For any of the paths counted by $a$, the number of ways to insert the $d$ diagonal steps into the available $2n - 2d - p$ positions is

$$b = \binom{2n-d-p-1}{d}.$$
Now (10) follows by simplifying $ab$. \hfill \Box

We give the $q$-analog of Proposition 1 in Proposition 2. In the proofs of Proposition 2 and Theorem 3, we use the basic hypergeometric function

$$
\Phi_1\left( \frac{a}{q}, \frac{b}{q} ; \frac{c}{q} ; z \right) := \sum_{n \geq 0} \frac{(q^n a)_n (q^n b)_n}{(q^n c)_n} z^n ,
$$

where $(q^n a)_n = (q^n; q)_n := (1 - q^n)(1 - q^{a+1}) \cdots (1 - q^{a+n-1})$ and the $q$-analog of Vandermonde’s Theorem \cite{[?, AIV.1]},

$$
\Phi_1\left( \frac{q^{-n}}{q}, \frac{q^n}{q} ; \frac{q^n}{q} ; q \right) = \frac{(q^{c-a})_n q^{an}}{(q^n)_n} . \tag{11}
$$

For ease of notation, we sometimes write $[n]$ for $[n]_q$.

**Proposition 2** For all $n, p \in \mathbb{Z}_{>0}$ and all $d \in \mathbb{Z}_{>0}$ such that $n \geq d + p$,

$$
q^{\left( \frac{n}{2} \right) - \left( \frac{d}{2} \right) - \left( \frac{p}{2} \right) - \left( \frac{1}{2} \right)} S_{n,d,p}(q, 1/q) = \left[ \begin{array}{c} p \end{array} \right]_q \left[ \begin{array}{c} 2n - d - p \end{array} \right] \left[ \begin{array}{c} n - d - p, n - d, d \end{array} \right]_q . \tag{12}
$$

**Proof.** We argue by induction on $n$. First observe that when $n = 1$ we must have $p = 1$ and $d = 0$, in which case it is routine to verify that both sides of (12) are equal to 1.

Suppose $n > 1$. It is routine to verify that

$$
S_{n,d,n-d}(q, 1/q) = q^{\left( \frac{n}{2} \right) - \left( \frac{d}{2} \right) - \left( \frac{1}{2} \right)} \binom{n-1}{d} ,
$$

so (12) holds when $n = d + p$. Therefore we may assume $n > d + p$. Now multiply both sides of (8) by $q^{\left( \frac{n}{2} \right) - \left( \frac{d}{2} \right) - \left( \frac{1}{2} \right)} = q^{\left( \frac{n}{2} \right) - \left( \frac{d}{2} \right) - \left( \frac{1}{2} \right)}$, set $t = 1/q$, and reverse the order of summation to find

$$
q^{\left( \frac{n}{2} \right) - \left( \frac{d}{2} \right) - \left( \frac{1}{2} \right)} S_{n,d,p}(q, 1/q) = \sum_{m=0}^{d} \sum_{r=0}^{n-d} \left[ \begin{array}{c} r + p + m - 1 \end{array} \right] \left[ \begin{array}{c} r - p - 1, m \end{array} \right] q^{\left( \frac{n}{2} \right) - \left( \frac{d}{2} \right) - \left( \frac{1}{2} \right)} \cdot
$$

Observe that those terms for which $r = 0$ in this last line are all equal to zero unless $n = d + p$. Therefore we may use induction to obtain

$$
q^{\left( \frac{n}{2} \right) - \left( \frac{d}{2} \right) - \left( \frac{1}{2} \right)} S_{n,d,p}(q, 1/q) = \sum_{m=0}^{d} \sum_{r=1}^{n-p-d} \frac{[r + p + m - 1][r][2n - 2p - m - d - r - 1]q^{r-1}(n-m-p)}{[r][m][p-1][n-p-d-r][n-p-d][d-m]!}.
$$

Therefore we may use induction to obtain

$$
= \sum_{m=0}^{d} \sum_{r=1}^{n-p-d} \frac{[r + p + m + (r-1)][r][2n - 2p - m - d - 2 - (r-1)][n-p-d-1-(r-1)]!}{[r-1][n-p-d-1-(r-1)]!} q^{r-1}(n-m-p),
$$

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where \( \text{pow} := \binom{n+1}{2} - n \binom{n}{2} - \binom{n-p-m}{2} - \binom{d-m}{2} \). Now set \( u = r - 1 \) and write the second sum in terms of the basic hypergeometric function \( _2\Phi_1 \):

\[
_q\Phi_r(q^{-1}) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(q^n; q)_n} \frac{1}{(1; q)_n}.
\]

employing the simple identity

\[
(q^a; q)_m r = \frac{(q^a; q)_m q^{(r+1)(m+a)}/(1-m-a; q)_r}{(q^{1-m-a}; q)_r},
\]

and obtaining

\[
_q\Phi_r(q^{-1}) S_{n,d,p}(q,1/q) = \frac{1}{[p-1]!\{n-p-d\}!} \sum_{m=0}^{d} \frac{[p+m]!\{2n-2p-d-m-2\}!q^{\text{pow}}}{[m]!\{d-m\}!\{n-p-d-1\}!} \sum_{u=0}^{\infty} \frac{(q^{p+m+1})_u (q^{-1})_u}{(q)_u (q^{-2n-2p-d-m-2})_u} q^{u(n-p-d-1)-(2n-2p-d-m-2)} q^{u(n-m-p)}.
\]

Since the exponent \( u((n-p-d-1) - (2n-2p-d-m-2)) + u(n-m-p) \) is equal to \( u \), (11) implies that

\[
\sum_{u=0}^{\infty} \frac{(q^{p+m+1})_u (q^{-1})_u}{(q)_u (q^{-2n-2p-d-m-2})_u} q^u
\]

\[
eq \frac{(q^{-1})_u}{(q^{-2n-2p-d-m-2})_u} q^{(p+m+1)(n-p-d-1)} \cdot \frac{(q^{2n-p-d-1})_{n-p-d-1} \cdots (q^{n+1-1}) q^{-(p+m+1)(n-p-d-1)+(p+m+1)(n-p-d-1)}}{(q^{2n-2p-d-m-2})_{n-p-d-1} \cdots (q^{n-p-m-1})}
\]

\[
= \frac{[2n-p-d-1]!\{n-p-d-1\}}{[n]!\{2n-2p-d-m-2\}!}.
\]

Therefore \( _2\Phi_1(q^{-1}) S_{n,d,p}(q,1/q) \) reduces to

\[
\frac{[2n-p-d-1]!\{n-p-1\}}{[p-1]!\{n-p-d\}!\{n-p-d-1\}!\{n\}!} \sum_{m=0}^{d} \frac{[p+m]!}{[m]!\{d-m\}!} \sum_{m=0}^{d} \frac{q^{\text{pow}}}{(q)_m \{q^{-n-p+1}\}^m q^{m(d-(n-p-1))}}.
\]
Since \( \text{pow} + m(d - n + p + 1) = n(p - 1) + m \), the final sum can be written in terms of \( q^{n(p-1)} \) \( 2\Phi_1 \left( \begin{array}{c} q^{-d}, q^{p+1} \\ q^{-n+p+1} \\ q \end{array} ; q \right) \). From (11),

\[
\sum_{m=0}^{d} \frac{(q^{-d})_m (q^{p+1})_m}{(q)_m (q^{-n+p+1})_m} q^m = \frac{(q^{-n})_d (q^{p+1})_d}{(q^{1+p-n})_d} = \frac{(q^n - 1) \cdots (q^{n-d+1} - 1)}{(q^n - p - 1) \cdots (q^{n-p-d+1} - 1)} q^{-(p+1)d + (p+1)d} = \frac{[n]![n-p-d-1]!}{[n-d]![n-1-p]!}.
\]

Simplify this last line to obtain

\[
q^{\binom{n}{2} - \binom{d}{2}} S_{n,d,p}(q, 1/q) = q^{n(p-1)} \frac{[p][2n-p-d-1]!}{[n-p-d]![d]![n-d]!},
\]
as desired. \( \square \)

We now use Proposition 2 and (9) to evaluate \( q^{(n+d-1)(n-d)/2} S_{n,d}(q, 1/q) \).

**Theorem 3** For all \( n, d \in \mathbb{Z}^\geq 0 \) such that \( n \geq d \), we have

\[
q^{(n+d-1)(n-d)/2} S_{n,d}(q, 1/q) = \frac{1}{[n-d+1]} \binom{2(n-d)}{n-d} \frac{[2n-d]}{q}.
\]

and

\[
q^{(n+d-1)(n-d)/2} S_{n,d}(q, 1/q) = \frac{[2n-d]!}{[n-d+1][d]![n-d]!}.
\]

**Proof.** Observe that (13) and (14) are equivalent, so it is sufficient to prove (14). To do this, we argue by induction on \( n \). Observe that if \( n = 0 \) then \( d = 0 \) and both sides of (14) are equal to 1. Also observe that if \( n = 1 \) then \( d = 0 \) or \( d = 1 \), and in either case both sides of (14) are equal to 1.

Suppose \( n > 1 \) and abbreviate \( S = q^{(n+d-1)(n-d)/2} S_{n,d}(q, 1/q) \). Multiply both sides of (9) by \( q^{\frac{n}{2} - \frac{d}{2}} = q^{\frac{(n-d)(n+d-1)}{2}} \) and set \( t = 1/q \) to find that \( S \) is equal to

\[
\sum_{k=0}^{d} \sum_{p=1}^{n-d} \frac{[p+k]!}{[p]![k]!} q^{\frac{(n-k)}{2} - \frac{(d-k)}{2} - (n-k)(p-1)} S_{n-k,d-k,p}(q, 1/q) q^{\frac{(n)}{2} - \frac{(d)}{2} - (n-k) + (d-k) + (n-k)(p-1)}.
\]

Now use Proposition 2 to simplify this expression, finding that \( S \) is equal to

\[
\sum_{k=0}^{d} \sum_{p=1}^{n-d} \frac{[p+k]!}{[p]![k]!} \frac{[p][2n-k-d-p-1]!}{[n-d-p]![n-d]![d-k]!} q^{\frac{(n)}{2} - \frac{(d)}{2} - (n-k) + (d-k) + (n-k)(p-1)}.
\]

\[
= \sum_{k=0}^{d} \frac{q^{\frac{(n)}{2} - \frac{(d)}{2} - (n-k) + (d-k)}}{[k]![n-d]![d-k]!} \sum_{p \geq 1} \frac{[k+1+(p-1)]![2n-d-k-2-(p-1)]!}{[p-1]![n-d-1-(p-1)]!} q^{(p-1)(n-k)}.
\]
Now set \( u = p - 1 \) and write the inner sum in terms of \( _2\Phi_1 \)

\[
\left( \frac{q^{k+2}, q^{-(n-d-1)}}{q^{-(2n-d-k-2)} ; q} \right)
\]
to find that

\[
S = \frac{1}{[n-d]!} \sum_{k=0}^{d} \frac{q^{\binom{n-k}{2} - \binom{d}{2} + \binom{d-k}{2}}\cdot [k+1]!\cdot [2n-d-k-2]!}{[k]!\cdot [d-k]!\cdot [n-d-1]!} \times \sum_{u \geq 0} \frac{(q^{k+2})_u (q^{-(n-d-1)})_u}{(q)_u (q^{-(2n-d-k-2)})_u} q^{u(u-n) + u(n-d-1-2n+d+k+2)}.
\]

Now observe that by (11),

\[
\sum_{u \geq 0} \frac{(q^{k+2})_u (q^{-(n-d-1)})_u}{(q)_u (q^{-(2n-d-k-2)})_u} q^{u} = \frac{(q^{-(2n-d)})_{n-d-1}}{(q^{-(2n-d-k-2)})_{n-d-1}} q^{(k+2)(n-d-1)}
\]

\[
= (q^{2n-d-1}) \cdots (q^{n+2} - 1) \cdots (q^{n-k} - 1)
\]

\[
= \frac{[2n-d]!\cdot [n-k-1]!}{[n+1]!\cdot [2n-d-k-2]!}.
\]

Now rewrite the sum indexed by \( k \) in terms of \( _2\Phi_1 \)

\[
\left( \frac{q^{-d}, q^2}{q^{1-n} ; q} \right)
\]
to obtain

\[
S = \frac{[2n-d]!}{[n+1]!\cdot [n-d]!\cdot [n-d-1]!} \sum_{k=0}^{d} \frac{q^{\binom{n-k}{2} - \binom{d}{2} + \binom{d-k}{2}}\cdot [k+1]!\cdot [n-1-k]!}{[k]!\cdot [d-k]!}.
\]

\[
= \frac{[2n-d]!\cdot [n-1]!}{[n+1]!\cdot [n-d]!\cdot [n-d-1]!\cdot [d]!} \sum_{k=0}^{d} \frac{(q^2)_k (q^{-d})_k}{(q)_k (q^{1-n})_k} q^{\binom{n-k}{2} - \binom{d}{2} + \binom{d-k}{2}} k^{1-n+d}.
\]

The exponent \( \binom{n-k}{2} - \binom{d}{2} + \binom{d-k}{2} + (1-n+d) \) is equal to \( k \), so by (11) our sum is

\[
\sum_{k=0}^{d} \frac{(q^2)_k (q^{-d})_k}{(q)_k (q^{1-n})_k} q^{k} = \frac{(q^{-1-n})_d q^{2d}}{(q^{1-n})_d}
\]

\[
= \frac{(q^{n+1} - 1) \cdots (q^{n+2-d} - 1)}{(q^{n-1} - 1) \cdots (q^{n-d} - 1)}
\]

\[
= \frac{[n+1]!\cdot [n-d-1]!}{[n+1-d]!\cdot [n-1]!}.
\]

It follows that

\[
q^{(n+d-1)(n-d)/2} S_{n,d}(q, 1/q) = \frac{[2n-d]!}{[n-d]!\cdot [d]!\cdot [n+1-d]!},
\]
as desired. \( \square \)
4 A Schröder Generalization of Haiman’s Statistic

In this section we define a statistic \( \text{sinv} \) on the set of Schröder paths from \((0,0)\) to \((n,n)\) which have \( d \) diagonal steps. We show bijectively that

\[
\sum_{\Pi \in \mathcal{S}_{n,d}} q^{\text{sinv}(\Pi)} t^{\text{area}(\Pi)} = \sum_{\Pi \in \mathcal{S}_{n,d}} q^{\text{area}(\Pi)} t^{\text{smaj}(\Pi)}.
\]

Our proof extends a bijection defined on Catalan paths by Haglund and Loehr \[?\]. Definition 6 is an extension of their definition for the statistic \( \text{dinv} \) (first discovered by Haiman) defined on \( \mathcal{D}_n = \mathcal{S}_{n,0} \). If \( \Pi \) is a Catalan path, then \( \text{sinv}(\Pi) = \text{dinv}(\Pi) + n \).

**Definition 6** We assign to each row of a path \( \Pi \in \mathcal{S}_{n,d} \) a nonnegative integer \( \text{area}_i(\Pi) \):

- If the step in row \( i \) (numbered from 1 to \( n \) starting at the top) of \( \Pi \) is a vertical step, then \( \text{area}_i(\Pi) \) is the number of upper triangles in row \( i \) which contribute to \( \text{area}(\Pi) \).
- If the step in row \( i \) is a diagonal step, then \( \text{area}_i(\Pi) \) is 0.5 plus the number of upper triangles in row \( i \) which contribute to \( \text{area}(\Pi) \).

The sequence \( (\text{area}_i(\Pi))_{i=1}^n \) is called the area sequence of \( \Pi \). To each row in \( \Pi \) we assign an inversion number, \( \text{sinv}_i(\Pi) \) according to the rules:

- If \( \text{area}_i \) is an integer, then \( \text{sinv}_i(\Pi) = \# \{(i,j) \mid i < j \text{ and } \text{area}_j(\Pi) = \text{area}_i(\Pi) + 1\} \)
- otherwise, \( \text{sinv}_i(\Pi) = \# \{(i,j) \mid i < j \text{ and } \text{area}_j(\Pi) = \text{area}_i(\Pi) + 0.5\} \).

The inversion sequence of \( \Pi \) is \( (\text{sinv}_i(\Pi))_{i=1}^n \). Define \( \text{sinv}(\Pi) = \sum_{i=1}^n \text{sinv}_i(\Pi) \). Note that \( \text{area}(\Pi) = \sum_{i=1}^n \lfloor \text{area}_i(\Pi) \rfloor \).

Figure 5 gives an example of the area and \( \text{sinv} \) statistics for a Schröder path.

Haglund and Loehr gave a bijection on \( \mathcal{D}_n \) which switches the statistic area with \( \text{dmaj} \) and \( \text{dinv} \) with area. Extending Haglund and Loehr’s work to Schröder paths, for each \( \Pi \in \mathcal{S}_{n,d} \) we construct a path \( \phi(\Pi) \in \mathcal{S}_{n,d} \) so that \( \text{area}(\Pi) = \text{smaj}(\phi(\Pi)) \) and \( \text{sinv}(\Pi) = \text{area}(\phi(\Pi)) \). Moreover, our bijection \( \phi : \mathcal{S}_{n,d} \to \mathcal{S}_{n,d} \) will have the property that a diagonal at height \( h \) in \( \Pi \) will correspond to a diagonal with shift \( h \) in \( \phi(\Pi) \). If the \( i \)th step in \( \Pi \) is a diagonal step, it will correspond to a diagonal step in \( \phi(\Pi) \) with height equal to \( \text{sinv}_i(\Pi) \).

**Theorem 4** For all \( n,d \in \mathbb{Z}^{\geq 0} \) with \( n \geq d \), there exists a constructive bijection \( \phi : \mathcal{S}_{n,d} \to \mathcal{S}_{n,d} \) such that for all \( \Pi \in \mathcal{S}_{n,d} \), we have \( \text{area}(\Pi) = \text{smaj}(\phi(\Pi)) \) and \( \text{sinv}(\Pi) = \text{area}(\phi(\Pi)) \). Therefore,

\[
\sum_{\Pi \in \mathcal{S}_{n,d}} q^{\text{sinv}(\Pi)} t^{\text{area}(\Pi)} = \sum_{\Pi \in \mathcal{S}_{n,d}} q^{\text{area}(\Pi)} t^{\text{smaj}(\Pi)}.
\]

**Proof.** Let \( \Pi \in \mathcal{S}_{n,d} \) have area sequence \( (\text{area}_i(\Pi))_{i=1}^n \) and inversion sequence \( (\text{sinv}_i(\Pi))_{i=1}^n \). Observe that \( \Pi \) has area partition \( \alpha(\Pi) =< 1^{a_1} 2^{a_2} \cdots k^{a_k} > \), where we write \( a_t = \# \{ \text{area}_i(\Pi) \mid \text{area}_i(\Pi) = t \} \). The height vector \( h = (h_0, h_1, \ldots, h_k) \) of \( \Pi \) is defined by
$h_t = \# \{ \text{area}_i(\Pi) | \text{area}_i(\Pi) = t + 0.5 \}$. Beginning at $(0,0)$, we construct $\phi(\Pi)$ by defining north, east, and diagonal steps so that $\phi(\Pi)$ will have balanced path $\beta = (\beta_1, \ldots, \beta_k, 0, 0) = (a_1 + a_2 + \cdots + a_k, a_2 + \cdots + a_k - 1 + a_k, \ldots, a_k, 0, 0) = \alpha'$ and shift vector $s = (s_0, s_1, \ldots, s_k) = (h_0, h_1, \ldots, h_k)$. Algorithm $[(\text{area}, \text{sinv}) \rightarrow (\text{smaj}, \text{area})]$:

Initialize to $(0,0)$.
Input: the area sequence $(\text{area}_i(\Pi))_{i=1}^{n}$.
Output: a path $\phi(\Pi) \in S_{n,d}$.
For $t = k$ to 0;
  For $i = 1$ to $n$;
    If $\text{area}_i(\Pi) = t$ then take a north step;
    If $\text{area}_i(\Pi) = t + 0.5$ then take a diagonal step;
    If $\text{area}_i(\Pi) = t + 1$ then take an east step;
  repeat;
repeat;

In Figure 6 we have the image of the path in Figure 5 under the map described above.

For each $t$, this algorithm constructs a path which begins at $(x, y) = ((a_{t+2} + \cdots + a_k) + (h_{t+1} + h_{t+2} + \cdots + h_k), (a_{t+1} + \cdots + a_k) + (h_{t+1} + h_{t+2} + \cdots + h_k))$ and takes $a_t$ north steps, $h_t$ diagonal steps, and $a_{t+1}$ east steps. If a given sequence is the area sequence for some Catalan path, then it must satisfy the condition: if $\text{area}_i(\Pi) > t > 0$ for some $i$, then $\text{area}_j(\Pi) = t$ for some $j > i$. Thus, for fixed $t$ our algorithm ends in a north step at $((a_{t+1} + a_{t+2} + \cdots + a_k) + (h_t + h_{t+1} + h_{t+2} + \cdots + h_k), (a_t + a_{t+1} + \cdots + a_k) + (h_t + h_{t+1} + h_{t+2} + \cdots + h_k))$. Our construction ensures that if a diagonal step in $\Pi$ has height $t$, then
the corresponding diagonal in $\phi(\Pi)$ will be inserted below and to the left of exactly $t$ NE corners, so the shift vector of $\phi(\Pi)$ is equal to the height vector of $\Pi$. Since the balanced path $\beta(\phi(\Pi))$ is defined on the Catalan path obtained by removing the diagonal steps from $\phi(\Pi)$, $\beta(\phi(\Pi)) = \alpha'$. As in the proof of Theorem 1, we get $area(\Pi) = smaj(\phi(\Pi))$. Note that $area(\phi(\Pi)) = area(\beta(\phi(\Pi))) + [area(\phi(\Pi)) - area(\beta(\phi(\Pi)))]$. To see that $area(\phi(\Pi)) = sinv(\Pi)$, write $sinv(\Pi)$ as the sum of the area under the balanced path $\beta(\phi(\Pi))$ plus the area between $\phi(\Pi)$ and $\beta(\phi(\Pi))$.

$$sinv(\Pi) = \sum_{i=1}^{n} \# \{(i, j) \mid i \leq j \text{ and } area_j(\Pi) = area_i(\Pi)\}$$

$$+ \sum_{i=1}^{n} \# \{(i, j) \mid i < j \text{ and } area_j(\Pi) = area_i(\Pi) + 1\}$$

$$+ \sum_{i=1}^{n} \# \{(i, j) \mid i < j \text{ and } area_j(\Pi) = area_i(\Pi) + 0.5\}.$$ 

Since $a_t = \# \{area_i(\Pi) \mid area_i(\Pi) = t\}$, the first group of terms in this sum becomes

$$\sum_{i=1}^{n} \# \{(i, j) \mid i \leq j \text{ and } area_j(\Pi) = area_i(\Pi)\} = \sum_{t=1}^{k} (a_t + a_t - 1 + a_t - 2 + \cdots + 2 + 1)$$

$$= \sum_{t=1}^{k} \binom{a_t + 1}{2}$$

$$= area(\beta(\phi(\Pi))).$$

Figure 6: The path $\phi(\Pi)$ for the path in Figure 5
If \((\phi(\Pi))'\) is the Catalan path associated with \(\phi(\Pi)\), then the second group of terms is equal to \([\text{area}(\phi(\Pi))') - \text{area}(\beta(\phi(\Pi)))\]. To see this note that for fixed \(t\), \((\phi(\Pi))'\) is constructed within the rectangle bounded below by \(a_{t+1}\) east steps of the balanced path \(\beta(\phi(\Pi))\) and on the right by \(a_t\) north steps of \(\beta(\phi(\Pi))\). The number of east steps which are inserted after each north step in this rectangle is \(#\{(i, j)\mid i < j; \text{area}_i(\Pi) = t; \text{area}_j(\Pi) = t + 1\}\), and

\[
\sum_{i=1}^{n} \#\{(i, j)\mid i < j \text{ and } \text{area}_j(\Pi) = \text{area}_i(\Pi) + 1\} = \sum_{t=1}^{k} \#\{(i, j)\mid i < j; \text{area}_i(\Pi) = t; \text{area}_j(\Pi) = t + 1\}.
\]

Finally,

\[
\sum_{i=1}^{n} \#\{(i, j)\mid i < j \text{ and } \text{area}_j(\Pi) = \text{area}_i(\Pi) + 0.5\}
\]

\[
= \sum_{t=0}^{k} \#\{(i, j)\mid i < j; \text{area}_i(\Pi) = t \text{ and } \text{area}_j(\Pi) = t + 0.5\}
\]

\[
+ \sum_{t=0}^{k} \#\{(i, j)\mid i < j; \text{area}_i(\Pi) = t + 1/2 \text{ and } \text{area}_j(\Pi) = t + 1\}
\]

counts the number of diagonal steps which follow each north step plus the number of east steps which follow each diagonal step. This sum represents the contribution to \([\text{area}(\phi(\Pi))') - \text{area}(\beta(\phi(\Pi)))\] which lies to the right and below the diagonal steps in \(\phi(\Pi)\).

To show Algorithm \([(\text{area}, \text{sinv}) \rightarrow (\text{smaj}, \text{area})]\) is bijective, we describe its inverse. Given a path \(\Pi \in S_{n,d}\), with balanced path \(\beta(\Pi)\) and shift vector \((s_0, s_1, \ldots, s_k)\), we construct an area sequence by inserting subsequences whose entries are \(t\), \(t + 0.5\), and \(t + 1\) for \(t = 0 \) to \(k\).

{
Algorithm \([(\text{smaj}, \text{area}) \rightarrow (\text{area}, \text{sinv})]\):

Initialize to \((a) = (0)\).
Input: a path \(\Pi \in S_{n,d}\) with balanced path \(\beta = (n, \beta_1, \ldots, \beta_k, 0, 0)\) and shift vector \((s_0, s_1, \ldots, s_k)\).
Output: an area sequence \((\text{area}_i(\phi^{-1}(\Pi)))_{i=1}^{n}\).
For \(t = 0 \) to \(k\):
Number \(\beta_t - \beta_{t+2} + 1\) steps of \(\Pi\) beginning at 
\((\beta_{t+1} + s_{t+1} + \cdots + s_k, \beta_t + s_{t+1} + \cdots + s_k)\), moving up the path.
Given the sequence \((a)\) created thus far, we insert a new subsequence of \(t + 0.5's\) ad \(t's\) moving left to right.
For \(i' = 1\) to \(\beta_t - \beta_{t+2} + s_t\):
If step \(i'\) is a north step then move past the next \(t\) in \((a)\);
If step \( i' \) is a diagonal step then insert \( t + 0.5 \)

in the position immediately to the left of the next \( t \) in (a);
If step \( i' \) is an east step then insert \( t + 1 \)

in the position immediately to the left of the next \( t \) in (a);
repeat;
repeat;

The final sequence (a) created by this algorithm is an area sequence \((\text{area}_1(\phi^{-1}(\Pi)))_{i=0}^n\).

In the input path, if a diagonal step or an east step occurs for \( t > 0 \), then there must be a north step below it, so the sequence produced satisfies: If \(\text{area}_i(\Pi) > t > 0 \) for some \( i \), then \(\text{area}_j(\Pi) = t \) for some \( j > i \).

We claim that Algorithm \([(\text{area}, \text{sinv}) \rightarrow (\text{smaj}, \text{area})]\) and Algorithm \([(\text{smaj}, \text{area}) \rightarrow (\text{area}, \text{sinv})]\) reverse one another. That is, \( \phi^{-1}(\phi(\Pi)) = \Pi \) and \( \phi(\phi^{-1}(\Pi)) = \Pi \). If \( \Pi \) has area partition \( \alpha(\Pi) \) and height vector \( h(\Pi) \), then \( \phi(\Pi) \) has balanced path \( \beta = \alpha' \) and shift vector \( s = h \). For each \( t \), \( \phi(\Pi) \) takes \( a_{t+1} \) north steps and \( h_t \) diagonal steps, to which Algorithm \([ (\text{smaj}, \text{area}) \rightarrow (\text{area}, \text{sinv}) ] \) assigns rows in \( \phi^{-1}(\phi(\Pi)) \) having \( \text{area}_{t}(\Pi) = t \) and \( \text{area}_{a_t}(\Pi) = t + 0.5 \), respectively. Therefore, \( \phi^{-1}(\phi(\Pi)) \) has area partition \( \alpha(\Pi) \) and height vector \( h(\Pi) \). The order in which the steps are taken determines the ordering of the rows in \( \phi^{-1}(\phi(\Pi)) \). Since for fixed \( t \), the algorithms reverse one another, we get \( \phi^{-1}(\phi(\Pi)) = \Pi \). Similarly, \( \phi(\phi^{-1}(\Pi)) = \Pi \).

\( \square \)

5 Conjectures and Open Problems

1. In Conjecture 1, we conjecture that for all \( n, d \geq 0 \), \( S_{n,d}(q,t) = \text{S}_{n,d}(t,q) \). Ideally, a proof would supply an involution \( \omega : S_{n,d} \rightarrow S_{n,d} \) such that \( \text{area}(\Pi) = \text{dmaj}(\omega(\Pi)) \)

for all \( \Pi \in S_{n,d} \). Unfortunately, \( S_{n,d}(q,t) \) restricted to a fixed balanced path \( \beta \) is not equal to the corresponding polynomial \( S_{n,d}(t,q) \) on paths having area partition \( \alpha = \beta' \). For example, when \( \beta = (5,3,1) \) and \( \alpha = <1^{2}2^{1}3^{1} > \), we have \( C_n(q,t) = S_{n,0}(q,t) |_{\beta = q^{7}t^{9} + 2q^{8}t^{9} + 2q^{9}t^{9} + q^{10}t^{9}, \text{while } C_n(q,t) = S_{n,0}(q,t) |_{\alpha = t^{7}q^{9} + 2t^{8}q^{9} + t^{9}q^{9} + t^{10}q^{9} + t^{11}q^{9}, \text{This is demonstrated in Figure 7 and Figure 8 where all of the corresponding paths are drawn, but the polynomials are not equal.}}

2. Garsia and Haiman extended \( C_n(q,t) \) and \( \mathcal{F}_n(q,t) \) for each positive integer \( m \):

\[
C_n^{(m)}(q,t) = \sum_{\mu + n} q^{(m+1)} \frac{t^{(m+1)}}{a(1-t)(1-q)} \prod_{l=0}^{(0,0)} (1 - q^{a+l}) \sum_{l} q^{a+l} t^{l}
\]

\[
\mathcal{F}_n^{(m)}(q,t) = \sum_{\mu + n} H_{\mu}[X; q,t] \frac{t^{(m+1)}}{a(1-t)(1-q)} \prod_{l=0}^{(0,0)} (1 - q^{a+l}) \sum_{l} q^{a+l} t^{l}
\]

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and showed that

\[ q^{m(n)} C_n^{(m)}(q, 1/q) = \frac{1}{[mn + 1]}_q [\binom{mn + n}{n}]_q \]

and

\[ C_n^{(m)}(q, 1) = \sum_{\Pi \in \mathcal{D}_n^{(m)}} q^{\text{area}(\Pi)}, \]

where \( \mathcal{D}_n^{(m)} \) is the set of lattice paths from \((0, 0)\) to \((mn, n)\) allowing north and east steps, and staying above the line \( y = \frac{1}{m}x \).

Haiman [?] has a conjectured, natural way of extending the definition of \( \text{dinv} \) to Dyck paths from \((0, 0)\) to \((n, n)\) to generate \( C_n^{(m)}(q, t) \), and N. Loehr [?] has found a natural extension of \( \text{dmaj} \) which also appears to generate these polynomials.
there natural extensions of the statistics $smaj$ and $sinv$ to Schröder paths from $(0,0)$ to $(mn,n)$ which have interesting polynomial properties?

3. The definition of $dinv$ was originally communicated to Garsia and Haglund by Haiman in terms of arm and leg length. We translate that definition to Schröder paths as follows.

A cell $c$ of a Schröder path $Π ∈ S_{n,d}$ is a lattice square which lies above and to the left of $Π$, to the right of the line $x = 0$, and below the line $y = n$. The arm of $c$ is the set of full cells to the right of $c$ and left of $Π$. We will write $a_c$ to denote the number of full cells in the arm of $c$. If $c$ lies above a horizontal step, the leg of $c$ is the set of full cells below $c$ and above $Π$. If $c$ lies above a diagonal step, we also include the half cell above the diagonal step. Denote the number of cells in the leg of $c$ by $l_c$, where $l_c$ may be fractional. Let $cv$ denote a cell which lies to the left of a vertical step in $Π$ and $cd$ a cell which lies to the left of a diagonal step in $Π$. Then

$$sinv(Π) = (n - d) + \#\{c = cv| l_c \leq a_c \leq l_c + 1\} + \#\{c = cd| a_c = l_c\}$$

See Figure 12.

![Figure 9: Arm and leg version of $dinv$ extended to Schröder paths](image)

Building on these ideas, it would be quite interesting to find a definition of $S_{n,d}(q,t)$ as a sum of rational functions analogous to Garsia and Haiman’s definition of $C_n(q,t)$. 

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