Problem 1 (10 pts)  Find where the following function is decreasing.

\[ f(x) = x^3 + 15x^2 - 600x + 50 \]

We take the first derivative:

\[ f'(x) = 3x^2 + 30x - 600 \]

To find its critical points, we set this equal to zero:

\[ 3x^2 + 30x - 600 = 0 \]

and solve:

\[
\begin{align*}
3x^2 + 30x - 600 &= 0 \\
3(x^2 + 10x - 200) &= 0 \\
3(x - 10)(x + 20) &= 0
\end{align*}
\]

and we see that the critical points are 10 and -20 (there are no points where \( f' \) is undefined).

On a number line we examine these points:

\[ \begin{array}{c}
\text{\(-20\)} & \text{\(-10\)} & \text{\(0\)} & \text{\(10\)} & \text{\(20\)}
\end{array} \]

We need to check \( f' \) in each of these intervals. We note

\[
\begin{align*}
f'(-100) &= 3(-100 - 10)(-100 + 20) > 0 \\
f'(0) &= -600 < 0 \\
f'(20) &= 3(20 - 10)(20 + 20) > 0
\end{align*}
\]

Since the sign of \( f' \) can only change at critical points, we can be sure that \( f' \) is only negative on the interval \((-20, 10)\). Thus, \( f \) is decreasing on \((-20, 10)\).

Problem 2 (10 pts)  For the previous problem, find where \( f(x) \) is concave up.

We now take the second derivative:

\[ f''(x) = 6x + 30 \]

This is only positive when

\[
\begin{align*}
6x + 30 &> 0 \\
6x &> -30 \\
x &> -5
\end{align*}
\]

So \( f \) is concave up when \( x > -5 \).

Problem 3 (5 pts)  Suppose \( f \) is a continuous function over each of the following domains. Specify for which ones \( f \) is guaranteed to have a global maximum.
Yes [2,10]

No (2,10)

No (−∞, ∞)

Yes −1 ≤ x ≤ 1

No x ≥ 0

This only happens on a closed finite interval. This is by the extreme value theorem.

Problem 4 (10 pts) Choose True or False:

FALSE Every function has a global maximum. Take x for instance.

FALSE Every function has a local maximum. Take x for instance.

TRUE If a point is on the endpoint of the domain of f, then f cannot have a local minimum there. To be a local minimum we need to check an open interval around the point, which is impossible if it occurs at the boundary.

TRUE If f is defined, continuous, and differentiable everywhere, and if f’ is never zero, then the global maximum for f on [0,3] must be at either 0 or at 3. This is a consequence of the extreme value theorem (which says that a global max must exist) and Fermat’s principle which says that the local maxima can only happen at critical points.

TRUE If f(x) is differentiable at x = 8, then f(8) + f’(8)·(x − 8) is a good approximation for f(x) when x is close to 8. This is the local linear approximation.

Problem 5 (5 pts) Suppose f(x) is a function that is defined, continuous, and differentiable everywhere. Suppose f(1) = 2 and f(10) = 7. What is the most that can be concluded by the Mean Value Theorem?

We first compute:

\[
\frac{f(10) − f(1)}{10 − 1} = \frac{7 − 2}{10 − 1} = \frac{5}{9}
\]

The Mean Value theorem says that there is a number c between 1 and 10 so that f’(c) = \frac{5}{9}.

Problem 6 (10 pts) If f(x) is a function and some of the values of f(x) and f’(x) are given in the tables below, use the local linear approximation to estimate f(30.2).

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>f’(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50</td>
<td>12</td>
</tr>
<tr>
<td>10</td>
<td>65</td>
<td>30</td>
</tr>
<tr>
<td>20</td>
<td>103</td>
<td>38</td>
</tr>
<tr>
<td>30</td>
<td>150</td>
<td>7</td>
</tr>
<tr>
<td>40</td>
<td>165</td>
<td>20</td>
</tr>
<tr>
<td>50</td>
<td>193</td>
<td>50</td>
</tr>
</tbody>
</table>
The closest value to 30.2 is 30, so we do a linear approximation at \( x = 30 \):

\[
f(30.2) = f(30) + f'(30)(30.2 - 30) \\
= 150 + 7 \cdot (0.2) \\
= 151.4
\]

**Problem 7 (15 pts)** What are the dimensions of the cylinder of smallest total surface area that holds \( 16\pi \) units of volume? You must justify your answer by convincing the reader that the answer really is the global minimum.

We are minimizing surface area, and the variables we control are the radius of the circle \( r \) and the height \( h \). We know the volume, which is \( \pi r^2 h \), is \( 16\pi \) units, so

\[
\pi r^2 h = 16\pi.
\]

We can use this to eliminate \( h \) as follows:

\[
h = \frac{16\pi}{\pi r^2} = 16r^{-2}.
\]

We are minimizing surface area, which contains two circles of radius \( r \) and the side which can be rolled like the wrapper of a can into a rectangle of side \( 2\pi r \) and width \( h \). So the surface area is

\[
A = 2\pi r^2 + 2\pi rh
\]

Eliminating \( h \) we get

\[
A(r) = 2\pi r^2 + 2\pi r(16r^{-2}) \\
= 2\pi r^2 + 32\pi r^{-1}
\]

To find critical points we first find where \( A' \) is zero:

\[
A'(r) = 4\pi r - 32\pi r^{-2} = 0
\]

and solve:

\[
4\pi r - 32\pi r^{-2} = 0 \\
4\pi r = 32\pi r^{-2} \\
4r = 32r^{-2} \\
r = 8r^{-2} \\
r = \frac{8}{r^2} \\
r = 2
\]

Now \( A'(r) \) is also not defined when \( r = 0 \), but \( A(r) \) is not defined there either, so this is not a critical point. Therefore \( r = 2 \) is the only critical point.

The possible domain for \( r \) is \( r > 0 \). We use the second derivative test:

\[
A''(r) = 4\pi + 64\pi r^{-3} > 0
\]
so we know that $r = 2$ is a local minimum. We also conclude from this that $A'(r)$ is negative immediately to the left of $r = 2$ and positive immediately to the right of $r = 2$. Since there is only one critical point and $A'(r)$ can only change sign at a critical point, we conclude that $A'(r)$ is negative for $0 < r < 2$ and positive for $r > 2$. Therefore, $r = 2$ is a global minimum.

At this point $h = \frac{16}{4} = \frac{16}{2} = 4$.
So the cylinder should have radius 2 units and height 4 units.

**Problem 8 (10 pts)** A certain crystal always grows in the shape of a cube. A certain specimen of this crystal which is 30mm on a side is growing, with side length currently growing at a rate of 3 mm per day. How quickly is the volume of the specimen growing at this instant?

Let $x$ be the side length of the cube and $V$ be the volume of the cube. Now

\[ V = x^3 \]

and we differentiate both sides with respect to $t$:

\[ \frac{dV}{dt} = 3x^2 \frac{dx}{dt}. \]

Currently $x = 30$ and $\frac{dx}{dt} = 3$, so

\[ \frac{dV}{dt} = 3(30)^2 \cdot 3 = 8100 \text{ mm}^3 \text{ per day} \]

**Problem 9 (5 pts)** Suppose the side length of a cube is measured to be 30.0 mm, with an experimental error of ±0.2 mm. Thus, the volume of the cube is $30^3 = 27000 \text{ mm}^3$. Find the error of this volume estimate.

Again, let $x$ be the side length and $V$ be the volume, with

\[ V = x^3. \]

We find the differential:

\[ dV = 3x^2 \, dx \]

and use $x = 30.0$, $dx = 0.2$. We find

\[ dV = 3(30.0)^2 \cdot 0.2 = 540 \]

so the error of this volume estimate is 540 mm$^3$.

**Problem 10 (5 pts)** Write down the second degree Taylor polynomial around $a = 1$ for

\[ f(x) = \tan^{-1}(x). \]

We compute the derivative and second derivative of the arctan function:

\[ f'(x) = \frac{1}{1 + x^2} = (1 + x^2)^{-1} \]

\[ f''(x) = -1(1 + x^2)^{-2}(2x) \]
And we plug in the value 1 into each function:

\[
\begin{align*}
  f(1) &= \frac{\pi}{4} \\
  f'(1) &= \frac{1}{1 + 1^2} = \frac{1}{2} \\
  f''(1) &= -1(1 + 1^2)^{-2} = -\frac{2}{2^2} = -\frac{1}{2}
\end{align*}
\]

Therefore the Taylor polynomial, which is

\[
f(1) + f'(1)(x - 1) + \frac{1}{2!} f''(1)(x - 1)^2
\]

is

\[
\frac{\pi}{4} + \frac{1}{2}(x - 1) - \frac{1}{4}(x - 1)^2.
\]

**Problem 11 (10 pts)** Find the following limit:

\[
\lim_{x \to 0} \frac{4x}{\tan(3x)}
\]

Plugging in \(x = 0\) we get \(0/0\), which is an indeterminate form for which we can use L’Hospital’s rule:

\[
\lim_{x \to 0} \frac{4}{\tan(3x)} = \lim_{x \to 0} \frac{\frac{d}{dx} (4x)}{\frac{d}{dx} (\tan(3x))} = \frac{4}{\sec^2(3x) \cdot 3} = \frac{4}{3}
\]

**Problem 12 (5 pts)** Find a function \(f(x)\) that has a tangent line of \(y = 3x + 70\) at \(x = 4\). Hint: They call it a local linear approximation for some reason. Draw a rectangle around your answer.

There are many possible answers to this one, but the easiest can be obtained from the following observation: the tangent line is an approximation to the function. What better approximation to a function than the function itself? Therefore the original function,

\[
f(x) = 3x + 70
\]

is a possible answer. You can check that it indeed has the right slope and value at \(x = 4\), but if you think about it, that part is obvious.

Actually any function that has \(f(4) = 82\) and \(f'(4) = 3\) will work, but the answer above is the easiest.