Math 104 Midterm 3 Answers

Problem 1 (5 pts) Solve for $x$ in degrees (to the nearest degree), if we assume $x$ is between $0^\circ$ and $90^\circ$.

$7\sin x + 1 = 4$

$7\sin x + 1 = 4$
$7\sin x = 3$
$\sin x = \frac{3}{7}$

$x = \sin^{-1}\left(\frac{3}{7}\right)$

$x = 25^\circ$

Problem 2 (10 pts) Solve for $x$ in degrees. Find all solutions.

$2\cos^2 x + \sin x = 2$

We use the identity

$\cos^2 x = 1 - \sin^2 x$

to turn this equation into

$2(1 - \sin^2 x) + \sin x = 2$

and simplify:

$2(1 - \sin^2 x) + \sin x = 2$
$2 - 2\sin^2 x + \sin x = 2$
$-2\sin^2 x + \sin x = 0$
$\sin x(-2\sin x + 1) = 0$

so that either

$\sin x = 0$

or

$-2\sin x + 1 = 0$

This, in turn is equivalent to

$\sin x = \frac{1}{2}$

For $\sin x = 0$ we get

$x = 360^\circ n$
where \( n \) is an integer, and for \( \sin x = 1/2 \) we get
\[
x = 30^\circ + 360^\circ n, \quad 150^\circ + 360^\circ n
\]
where \( n \) is an integer.

**Problem 3 (10 pts)** Solve this equation for \( y \).

\[
x = 5 \cos(3y + 5)
\]

\[
x = 5 \cos(3y + 5)
\]
\[
\frac{x}{5} = \cos(3y + 5)
\]
\[
\cos^{-1} \left( \frac{x}{5} \right) = 3y + 5
\]
\[
\cos^{-1} \left( \frac{x}{5} \right) - 5 = 3y
\]
\[
\frac{\cos^{-1} \left( \frac{x}{5} \right) - 5}{3} = y
\]

**Problem 4 (10 pts)** A triangle \( ABC \) has side lengths \( a = 10 \), \( b = 20 \), and \( c = 25 \). Solve the triangle, using two significant digits.

Using the law of cosines on the longest side \( c \), we get
\[
c^2 = a^2 + b^2 - 2ab \cos C
\]
which in our case gives us
\[
25^2 = 10^2 + 20^2 - 2 \cdot 10 \cdot 20 \cos C
\]
\[
625 = 100 + 400 - 400 \cos C
\]
\[
625 = 500 - 400 \cos C
\]
\[
625 - 500 = -400 \cos C
\]
\[
125 = -400 \cos C
\]
\[
\frac{125}{400} = \cos C
\]
\[
C = \cos^{-1} \left( \frac{125}{400} \right)
\]
\[
= \cos^{-1}(-.3125)
\]
\[
= 108.21^\circ
\]
\[
\approx 110^\circ
\]
We can use the law of sines to find angle $A$:

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

which in our case is

$$\frac{25}{\sin(110^\circ)} = \frac{10}{\sin A}$$

$$25 \cdot \sin A = 10 \cdot \sin(110^\circ)$$

$$26.60 \cdot \sin A = 10$$

$$\sin A = \frac{10}{26.60} = 0.3759$$

$$A = \sin^{-1}(0.3759) = 22.08^\circ \approx 22^\circ$$

Note here that $A$ cannot be obtuse since $C$ is already obtuse, and no triangle can have more than one obtuse angle.

We then know that

$$B = 180^\circ - (A + C) = 180^\circ - (110^\circ + 22^\circ) = 180^\circ - 132^\circ = 48^\circ$$

Problem 5 (5 pts) A triangle $ABC$ has $A = 20^\circ$, $b = 10$, and $a = 8$. Find all the possible angles $B$ to two significant digits.

This is an ASS triangle specification, so we need to be careful when using the law of sines. The law of sines says

$$\frac{8}{\sin 20^\circ} = \frac{10}{\sin B}$$

so that makes

$$\sin B = \frac{10 \sin 20^\circ}{8}$$

that is,

$$\sin B = 0.427525$$

We find the possible values of $B$ by taking

$$B = \sin^{-1}(0.427525) = 25^\circ$$

but also the other solution is $B = 180^\circ - 25^\circ = 155^\circ$, which to two significant digits is 160°.

Problem 6 (5 pts) For the triangle $ABC$ with angle $B = 30^\circ$ and side $c = 8$, consider the following values for the length of side $b$, and label them according to whether there is no, one, or two possible triangles $ABC$.

- $b = 1$ **None:** $b < c \sin B$
\[ b = 4 \quad \text{One: } b = c \sin B \]
\[ b = 6 \quad \text{Two: } c \sin B < b < c \]
\[ b = 7 \quad \text{Two: } c \sin B < b < c \]
\[ b = 9 \quad \text{One: } b \geq c \]

Problem 7 (5 pts) A triangle ABC has \( b = 60 \), \( c = 10 \), and \( A = 30^\circ \). Find the area of the triangle.

\[
\text{area} = \frac{1}{2} bc \sin A = \frac{1}{2} \cdot 60 \cdot 10 \cdot \sin 30^\circ = 150
\]

Problem 8 (5 pts) Given vectors \( \vec{a} = \langle 2, 3 \rangle \) and \( \vec{b} = \langle 5, -11 \rangle \), find the following:

\[
\begin{align*}
\vec{a} + \vec{b} & : \langle 7, -8 \rangle \\
\vec{a} - \vec{b} & : \langle -3, 14 \rangle \\
2\vec{a} & : \langle 4, 6 \rangle \\
\vec{a} \cdot \vec{b} & : 2 \cdot 5 + 3 \cdot (-11) = 10 - 33 = -23 \\
|\vec{a}| & : \sqrt{2^2 + 3^2} = \sqrt{13}
\end{align*}
\]

Problem 9 (5 pts) Given the vectors \( \vec{a} \) and \( \vec{b} \) shown, draw \( \vec{a} + \vec{b} \).

Problem 10 (5 pts) A certain vector is of magnitude 10 has a direction of 15 degrees up from the horizontal. Find the x and y coordinates of this vector, to two significant digits.

We use the formulas

\[
\begin{align*}
x &= r \cos \theta = 10 \cos 15^\circ \approx 9.659 \\
y &= r \sin \theta = 10 \sin 15^\circ \approx 2.588
\end{align*}
\]

So the coordinates are (2.6, 9.7) to two significant digits.

Problem 11 (10 pts) An airplane sets a bearing of 110 degrees from north, traveling at an airspeed of 500 miles per hour. There is a wind coming from the west at 40 miles per hour. Find the speed the plane is traveling with respect to the ground, to the nearest mile per hour.
The following diagram illustrates the situation. The airplane is at the upper left corner of the triangle, with air velocity given by the lower arrow, at a speed of 500 miles per hour. The wind velocity is the horizontal vector, and we add these to get the vector whose length is marked $x$. (Note: This picture is not to scale.) The bearing is 110°, which is marked in the upper left. This is the same angle as the one so marked at the bottom.

![Diagram with vectors and angles]

The triangle shown, then, has an unknown length $x$, whose opposite angle is $270° - 110° = 160°$. We therefore use the law of cosines to find $x$:

$$x^2 = 500^2 + 40^2 - 2 \cdot 500 \cdot 40 \cos 160°$$

We simplify somewhat:

$$x^2 = 250000 + 1600 - 40000 \cos 160°$$

and plug into a calculator to get

$$x^2 = 289187.7$$

so that

$$x = 537.76 \text{ miles per hour}$$

which to the nearest mile per hour is 538 miles per hour.

**Problem 12 (5 pts)** Write $-3.00 + 7.00i$ in polar form using degrees for the angle, to three significant digits.

We use $r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 7^2} = \sqrt{9 + 49} = \sqrt{58} \approx 7.62$. We also know that

$$\tan \theta = \frac{y}{x} = \frac{7}{-3} = -2.333$$

so that we can use arctangent to find

$$\theta = -66.80°$$

or

$$180° - 66.80° = 113.2°$$

Since $(-3, 7)$ is in the second quadrant, we must use $113.2°$.

Therefore we have

$$7.62 \text{ cis } 113°$$
Problem 13 (5 pts) Find

\[
\frac{20 \, \text{cis} \, 110^\circ}{4 \, \text{cis} \, 80^\circ}.
\]

We divide the \( r \): \( 20/4 = 5 \); and subtract the \( \theta \)s: \( 110^\circ - 80^\circ = 30^\circ \). So we get

\[
5 \, \text{cis} \, 30^\circ.
\]

Problem 14 (5 pts) Find

\[
(2 \, \text{cis} \, 40^\circ)^5.
\]

We raise \( r \) to the 5th power: \( 2^5 = 32 \); and multiply \( \theta \) by 5: \( 5 \cdot 40^\circ = 200^\circ \). So we have

\[
32 \, \text{cis} \, 200^\circ.
\]

Problem 15 (10 pts) Find all the fourth roots of

\[
16 \, \text{cis} \, 40^\circ.
\]

We take the 4th root of \( r \): \( \sqrt[4]{16} = 2 \); and divide \( \theta \) by 4: now in this case, \( \theta \) is not only \( 40^\circ \) but any multiple of \( 360^\circ \) added to it. So when we divide by 4, we get \( 40^\circ/4 = 10^\circ \) but also \( 100^\circ, 190^\circ, \) and \( 280^\circ \). So we have

\[
2 \, \text{cis} \, 10^\circ, 2 \, \text{cis} \, 100^\circ, 2 \, \text{cis} \, 190^\circ, 2 \, \text{cis} \, 280^\circ.
\]