Math 215 Practice Midterm 1 answers

**Problem 1 (5 pts)** Take all the first-order partial derivatives of the following functions:

\[ f(x, y) = x^2 e^{xy} \]

\[
\frac{\partial f}{\partial x} = 2xe^{xy} + x^2e^{xy}y \\
\frac{\partial f}{\partial y} = x^3e^{xy}
\]

\[ p(x, y) = xy + \ln y \]

\[
\frac{\partial p}{\partial x} = y \\
\frac{\partial p}{\partial y} = x + \frac{1}{y}
\]

**Problem 2 (5 pts)** Take all the first-order partial derivatives of the following functions:

\[ h(u, v) = \ln(u^2v + 5) \]

\[
\frac{\partial h}{\partial u} = \frac{1}{u^2v + 5}2uv \\
\frac{\partial h}{\partial v} = \frac{1}{u^2v + 5}u^2
\]

\[ g(x, y, z) = xy^z \]

\[
\frac{\partial g}{\partial x} = y^z \\
\frac{\partial g}{\partial y} = xzy^{z-1} \\
\frac{\partial g}{\partial z} = x(ln y)y^z
\]
Problem 3 (10 pts) Take all the first-order and second-order partial derivatives of the following function:

\[ U(x,t) = 10x^3t^5 \]

First order:

\[ \frac{\partial U}{\partial x} = 30x^2t^5 \]
\[ \frac{\partial U}{\partial t} = 50x^3t^4 \]

Second order:

\[ \frac{\partial^2 U}{\partial x^2} = 60xt^5 \]
\[ \frac{\partial^2 U}{\partial t^2} = 200x^3t^3 \]
\[ \frac{\partial^2 U}{\partial x\partial t} = 150x^2t^4 \]

Problem 4 (10 pts) The hardness \( H \) of a metal depends on the maximum temperature \( T \) in degrees Celsius during the production process and the purity \( P \). Suppose your metal production process has \( T = 8000 \) and \( P = 0.99 \). Also, suppose

\[ \frac{\partial H}{\partial T}(8000,0.99) = 0.096 \]

and

\[ \frac{\partial H}{\partial P}(8000,0.99) = 200. \]

Now suppose the temperature were to increase by 20 degrees Celsius, and the purity decreased by 0.01. What would be the approximate overall effect on the hardness?

By itself, the increase of temperature by 20 degrees contributes a change of hardness by

\[ 0.096 \cdot 20 = 1.92 \]

and the decrease in purity by 0.01 contributes a change of hardness by

\[ 200 \cdot (-0.01) = -2 \]

so altogether, we get a change of hardness of \( 1.92 + (-2) = -0.08 \) so that hardness decreases by 0.08.
**Problem 5 (5 pts)** Sketch the level set of \( f(x, y) = 2y + 4x \) at height 6.

We want the set of points \((x, y)\) so that \(2y + 4x = 6\). This is a line, in case you noticed. If not, it’s easy to see if you solve for \(y\), which eventually gives you

\[
y = 3 - 2x.
\]

This is a line of slope \(-2\) and \(y\)-intercept 3.

**Problem 6 (10 pts)** Find all critical points of the function

\[
f(x, y) = 3x^2 - 3xy + 6y + 23
\]

and determine which are relative maxima, relative minima, and saddles.

\[
\frac{\partial f}{\partial x} = 6x - 3y = 0
\]
\[
\frac{\partial f}{\partial y} = -3x + 6 = 0
\]

We solve the second equation and get

\[
-3x = -6
\]
\[
x = 2
\]

and plug it into the first equation and get

\[
6(2) - 3y = 0
\]
\[
12 = 3y
\]
\[
y = 4
\]
so that the only critical point is (2, 4).

We now apply the second derivative test:

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= 6 \\
\frac{\partial^2 f}{\partial y^2} &= 0 \\
\frac{\partial^2 f}{\partial x \partial y} &= -3 \\
D &= 6 \cdot 0 - (-3)^2 = 0 - 9 = -9 < 0
\end{align*}
\]

Therefore, (2, 4) is a saddle.

**Problem 7 (10 pts)** True or false:

**F** The partial derivative is so called because it is half a rate of change.

**T** As you follow along a level curve for a function \( f(x, y) \), the value of \( f \) will not change.

**F** If at a point (2, 4), \( f_x(2, 4) = 0 \), \( f_y(2, 4) = 0 \), and \( f_{xx}(2, 4) > 0 \), then we can conclude that (2, 4) is a relative minimum.

**T** For a constrained optimization problem, at the maximum along a constraint, the level curve will be tangent to the constraint.

**T** If \( p \) is the price of Ivory Soap, and \( q \) the price of Dove soap, and the demand for Ivory soap \( D(p, q) \) is viewed as a function of the price of Ivory and Dove, then we expect \( \frac{\partial D}{\partial q} > 0 \).

**Problem 8 (8 pts)** Suppose employee morale is a function \( M(p, q) \) where \( p \) is the pay rate and \( q \) is the desirability of the work they need to do. Interpret in words the following mathematical statement:

\[
\frac{\partial M}{\partial p} = 10
\]

Assuming the desirability of the work they need to do remains constant, every unit of increase in pay results in an increase of 10 units in employee morale.
Problem 9 (10 pts) Suppose a certain business produces two products: product A and product B. Let $x$ represent the price of product A and $y$ represent the price of product B, at which the business sells the product. Suppose the demand for product A is given by

$$1000 - x$$

and the demand for product B is given by

$$300 - y.$$  

The cost of running the business is

$$600,000 - 500x - 100y.$$  

Find the prices of the two products that maximize profit for the business.

We wish to maximize profit. The profit is revenue minus cost, and the revenue from product A is the price $x$ times the quantity sold, which is $1000 - x$. Similarly for product B. So the revenue is

$$x(1000 - x) + y(300 - y)$$

and the cost is

$$600,000 - 500x - 100y$$

so the profit is

$$P(x, y) = x(1000 - x) + y(300 - y) - (600,000 - 500x - 100y).$$

We expand this to

$$P(x, y) = 1000x - x^2 + 300y - y^2 - 600,000 + 500x + 100y$$

and simplify:

$$P(x, y) = 1500x - x^2 + 400y - y^2 - 600,000.$$  

Now take partial derivatives:

$$\frac{\partial P}{\partial x} = 1500 - 2x = 0$$

$$\frac{\partial P}{\partial y} = 400 - 2y = 0$$

By solving these equations we get

$$x = 750, y = 200.$$
The second derivatives are

\[
\begin{align*}
P_{xx} &= -2 \\
P_{yy} &= -2 \\
P_{xy} &= 0
\end{align*}
\]

\[
D = (-2)(-2) - 0^2 = 4 - 0 = 4 > 0
\]

\[
P_{xx} = -2 < 0
\]

So this is a relative maximum.

**Problem 10 (10 pts)** Find the absolute maximum of \(3x - 5y + 35\) subject to the constraint

\[
x^2 + y^2 = 34.
\]

We use the method of Lagrange multipliers:

\[
F(x, y, \lambda) = 3x - 5y + 35 + \lambda(x^2 + y^2 - 34)
\]

and take all partial derivatives and set them equal to zero:

\[
\begin{align*}
\frac{\partial F}{\partial x} &= 3 + \lambda(2x) = 0 \\
\frac{\partial F}{\partial y} &= -5 + \lambda(2y) = 0 \\
\frac{\partial F}{\partial \lambda} &= x^2 + y^2 - 34 = 0 \\
\lambda(2x) &= -3 \\
\lambda &= \frac{-3}{2x} \\
-5 + \left(\frac{-3}{2x}\right)(2y) &= 0 \\
\frac{3 \cdot 2y}{2x} &= 5 \\
-3 \cdot 2y &= 5 \cdot 2x \\
-6y &= 10x \\
y &= \frac{-10}{6}x \\
y &= \frac{5}{3}x
\end{align*}
\]
\[ x^2 + \left( -\frac{5}{3}x \right)^2 - 34 = 0 \]
\[ x^2 + \frac{25}{9}x^2 = 34 \]
\[ (1 + \frac{25}{9})x^2 = 34 \]
\[ \frac{34}{9}x^2 = 34 \]
\[ x^2 = \frac{9}{34} = 9 \]
\[ x = \pm 3 \]

When \( x = 3 \), \( y = -\frac{5}{3}(3) = -5 \). When \( x = -3 \), \( y = -\frac{5}{3}(-3) = 5 \). So the two possibilities are \((3, -5)\) and \((-3, 5)\). To find which is the actual absolute maximum, we plug these into the function we are trying to maximize: \(3x - 5y + 35\). For this, \((3, -5)\) gives \(3(3) - 5(-5) + 35 = 69\). For \((-3, 5)\), we have \(3(-3) - 5(5) + 35 = 1\). So the maximum value of the function occurs at \((3, -5)\).

**Problem 11 (5 pts)** The function \( f(x, y) \) has a critical point at \((2, 7)\). At this critical point, \( f_{xx}(2, 7) = 4 \) and \( f_{yy}(2, 7) = 30 \). Give one example of a value for \( f_{xy}(2, 7) \) for which \((2, 7)\) a saddle.

There are many possible answers here.
For a saddle, \( D < 0 \). That means
\[ 4 \cdot 30 - f_{xy}^2 < 0 \]
which can be written
\[ f_{xy}^2 > 120. \]
So \( f_{xy} \) could be 11, for instance. Many other possible answers are possible, also, as long as their square is larger than 120.

**Problem 12 (10 pts)** The function
\[ f(x, y) = x^3 + x^2 + y^2 + x^2y + xy \]
has a critical point at \((0, 0)\). Use the second derivative test to see what kind it is.
\[ f_x = 3x^2 + 2x + 2xy + y \]
\[ f_y = 2y + x^2 + x \]
\[ f_{xx} = 6x + 2 + 2y \]
\[ f_{xy} = 2x + 1 \]
\[ f_{yy} = 2 \]

At the point \((0, 0)\),

\[ f_{xx}(0, 0) = 2 \]
\[ f_{xy}(0, 0) = 1 \]
\[ f_{yy}(0, 0) = 2 \]

\[ D = 2 \cdot 2 - 1^2 = 4 - 1 = 3 > 0 \]

so that \((0, 0)\) is either a relative max or min.

\[ f_{xx}(0, 0) = 2 > 0 \]

so \((0, 0)\) is a relative minimum.