Math 510 HW #22 Answers

Problem 1 Ch. 9 Theo. ex. p. 484 #1a

$$\begin{aligned}
P(N(20)=2\,|\,N(60)=2) &= \frac{P(N(20)=2\cap N(60)=2)}{P(N(60)=2)}\\
&= \frac{P(N(20)=2)\cdot P(N(60)-N(20)=0)}{P(N(60)=2)}\\
&= \frac{\frac{(20\lambda)^2}{2!}e^{-20\lambda}\cdot\frac{1}{0!}e^{-40\lambda}}{\frac{(60\lambda)^2}{2!}e^{-60\lambda}}\\
&= \frac{\frac{400\lambda^2}{2!}e^{-60\lambda}}{\frac{3600\lambda^2}{2!}e^{-60\lambda}}\\
&= \frac{400}{3600}\\
&= \frac{1}{9}
\end{aligned}$$

Ch. 9 Theo. ex. p. 484 #1b

We first find the probability that no one arrived in the first 20 minutes:

$$\begin{aligned}
P(N(20)=0\,|\,N(60)=2) &= \frac{P(N(20)=0\cap N(60)=2)}{P(N(60)=2)}\\
&= \frac{P(N(20)=0)\cdot P(N(60)-N(20)=2)}{P(N(60)=2)}\\
&= \frac{\frac{1}{0!}e^{-20\lambda}\cdot\frac{(40\lambda)^2}{2!}e^{-40\lambda}}{\frac{(60\lambda)^2}{2!}e^{-60\lambda}}\\
&= \frac{\frac{1600\lambda^2}{2!}e^{-60\lambda}}{\frac{3600\lambda^2}{2!}e^{-60\lambda}}\\
&= \frac{1600}{3600}\\
&= \frac{4}{9}
\end{aligned}$$

Thus, the probability that at least one person arrived in the first 20 minutes is

$$1-\frac{4}{9}=\frac{5}{9}.$$

Ch. 9 Theo. ex. p. 484 #2

This means that in s seconds, there are no cars, with $\lambda = 3/60 = .05$:

$$P(N(s)=0)=\frac{(.05s)^0}{0!}e^{-.05s}=e^{-.05s}$$

For the given values

s	Prob. of surviving	
2	$e^{-.1}$	$= .9048$
5	$e^{-.25}$	$= .7788$
10	$e^{-.5}$	$= .6065$
20	e^{-1}	$= .3679$

Ch. 9 Theo. ex. p. 484 #3
In this case, we need $P(N(s) \leq 1)$, which is

$$P(N(s) \leq 1) = \frac{(.05s)^0}{0!}e^{-.05s} + \frac{(.05s)^1}{1!}e^{-.05s} = (1 + .05s)e^{-.05s}$$

For the given values,

s	Prob. of surviving	
5	$1.25e^{-.25}$	$= .9735$
10	$1.5e^{-.5}$	$= .9098$
20	$2e^{-1}$	$= .7358$
30	$2.5e^{-1.5}$	$= .5579$

Problem 2 Suppose $N(t)$ is a Poisson process with $\lambda = 2$. Find:
a. $P(N(2) = 3)$

$$\frac{(2 \cdot 2)^3}{3!}e^{-2\cdot 2} = .1954$$

b. $P(N(10) - N(7) = 4)$

$$= P(N(3) = 4) = \frac{(2 \cdot 3)^4}{4!}e^{-2\cdot 3} = .1339$$

c. $P(N(10) - N(7) = 4 \,|\, N(2) = 3)$

$$= P(N(3) = 4) = .1339$$

d. $P(N(5) - N(2) = 4 \,|\, N(3) = 1)$. **Hint: Split the event $N(3) = 1$ according to the possible values of $N(2)$.**
One approach:

$$\begin{aligned}
&= \frac{P(N(5) - N(2) = 4 \cap N(3) = 1)}{P(N(3) = 1)} \\
&= \frac{P(N(5) - N(2) = 4 \cap N(2) = 1 \cap N(3) - N(2) = 0) + P(N(5) - N(2) = 4 \cap N(2) = 0 \cap N(3) - N(2) = 1)}{P(N(3) = 1)} \\
&= \frac{P(N(5) - N(3) = 4 \cap N(2) = 1 \cap N(3) - N(2) = 0) + P(N(5) - N(3) = 3 \cap N(2) = 0 \cap N(3) - N(2) = 1)}{P(N(3) = 1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{P(N(5)-N(3)=4)P(N(2)=1)P(N(3)-N(2)=0)) + P(N(5)-N(3)=3)P(N(2)=0)\cap P(N(3)-N(2)=1)}{P(N(3)=1)} \\
&= \frac{\frac{(2\cdot 2)^4}{4!}e^{-2\cdot 2}\cdot\frac{(2\cdot 2)^1}{1!}e^{-2\cdot 2}\cdot\frac{(2\cdot 1)^0}{0!}e^{-2\cdot 1} + \frac{(2\cdot 2)^3}{3!}e^{-2\cdot 2}\cdot\frac{(2\cdot 2)^0}{0!}e^{-2\cdot 2}\cdot\frac{(2\cdot 1)^1}{1!}e^{-2\cdot 1}}{\frac{(2\cdot 3)^1}{1!}e^{-2\cdot 3}} \\
&= \frac{\frac{4^4\cdot 4}{4!}e^{-4-4-2} + \frac{4^3\cdot 2}{3!}e^{-4-4-2}}{6\,e^{-6}} \\
&= \frac{\frac{1024}{24}+\frac{128}{6}}{6}e^{-4} \\
&= \frac{1536}{144}e^{-4} = .1954
\end{aligned}
$$

Another approach:

$$
\begin{aligned}
&= P(N(5)-N(2)=4|N(3)=1\cap N(2)=0)P(N(2)=0|N(3)=1) \\
&\quad +P(N(5)-N(2)=4|N(3)=1\cap N(2)=1)P(N(2)=1|N(3)=1) \\
&= P(N(5)-N(3)=3|N(3)=1\cap N(2)=0)P(N(2)=0|N(3)=1) \\
&\quad +P(N(5)-N(3)=4|N(3)=1\cap N(2)=1)P(N(2)=1|N(3)=1) \\
&= P(N(5)-N(3)=3)P(N(2)=0|N(3)=1) + P(N(5)-N(3)=4)P(N(2)=1|N(3)=1) \\
&= \frac{4^3}{3!}e^{-4}\frac{1}{3} + \frac{4^4}{4!}e^{-4}\frac{2}{3} \\
&= .1954
\end{aligned}
$$

Problem 3 *Let $W(t)$ be a Brownian process so that $W(1)$ is a standard normal random variable. Suppose $S(t)$ is the value of an investment, so that at $t=0$, the value of the investment is \$100; and at $t=2$, the value of the investment is normally distributed with a mean of \$120 and standard deviation is \$40. If*

$$S(t) = a + bt + cW(t)$$

then find a, b, and c.

For $t=0$, $S(0)=100$, so

$$S(0) = a + b\cdot 0 + c(0) = a = 100$$

At $t=2$, $W(2)$ is normal with mean 0 and standard deviation $\sqrt{2}$; therefore, $S(2)$ will be normal with mean $100+2b$ and standard deviation $c\sqrt{2}$. If the mean is 120 and $\sigma=40$, then

$$
\begin{aligned}
100+2b &= 120 \\
c\sqrt{2} &= 40
\end{aligned}
$$

so that $a=100$, $b=10$, and $c=\frac{40}{\sqrt{2}}=28.284$

Problem 4 *For the previous problem, find the value of t so that we are 99% certain that $S(t)\geq \$150$.*

We start with

$$P(S(t) \geq 150) = .99$$

where $S(t)$ is normal with $\mu = 100 + 10t$ and $\sigma = 28.284\sqrt{t}$. The corresponding Z score is

$$P(Z \geq -2.33) = .99$$

so

$$\frac{150 - (100 + 10t)}{28.284\sqrt{t}} = -2.33$$

Solve for t:

$$\begin{aligned}
150 - 100 - 10t &= -2.33 \cdot 28.284\sqrt{t} \\
50 - 10t &= -65.9017\sqrt{t} \\
(50 - 10t)^2 &= 4343t \\
2500 - 1000t + 100t^2 &= 4343t \\
100t^2 - 5343t + 2500 &= 0 \\
t &= \frac{5343 \pm \sqrt{5343^2 - 4 \cdot 100 \cdot 2500}}{2 \cdot 100} \\
&= \frac{5343 \pm 5249}{200} \\
&= 52.96 \text{ or } .47
\end{aligned}$$

When we plug these into the original equation, it is apparent that $t = 52.96$ is the only solution ($t = .47$ gives a z score of $+2.33$).

Problem 5 *For the same investment, find* $P(S(5) \geq 160 \,|\, S(3) = 140)$. *Hint: rewrite* $S(t)$ *in terms of a Brownian process starting at* $t = 3$.

If we know that $S(3) = 140$, then $S(t)$ becomes a stock model with $t-3$, with an additional $140-100 = 40$: $S(t-3) + 40$. Therefore we need

$$P(S(2) + 40 \geq 160).$$

Since $S(2) + 40$ is a normal random variable with $\mu = 2 \cdot 10 + 140 = 160$ and $\sigma = 28.284 \cdot \sqrt{2}$, so in terms of Z, we need

$$P(Z \geq 0) = .5$$