Problem 1 Ch. 9 Theo. ex. p. 484 #1a

\[ P(N(20) = 2 \mid N(60) = 2) = \frac{P(N(20) = 2 \cap N(60) = 2)}{P(N(60) = 2)} = \frac{P(N(20) = 2) \cdot P(N(60) - N(20) = 0)}{P(N(60) = 2)} = \frac{(20\lambda)^2 e^{-20\lambda} \cdot \frac{1}{2!} e^{-40\lambda}}{(60\lambda)^2 e^{-60\lambda}} = \frac{400\lambda^2 e^{-60\lambda}}{3600\lambda^2 e^{-60\lambda}} = \frac{400}{3600} = \frac{1}{9}. \]

Ch. 9 Theo. ex. p. 484 #1b

We first find the probability that no one arrived in the first 20 minutes:

\[ P(N(20) = 0 \mid N(60) = 2) = \frac{P(N(20) = 0 \cap N(60) = 2)}{P(N(60) = 2)} = \frac{P(N(20) = 0) \cdot P(N(60) - N(20) = 2)}{P(N(60) = 2)} = \frac{\frac{1}{2!} e^{-20\lambda} \cdot (40\lambda)^2 e^{-40\lambda}}{(60\lambda)^2 e^{-60\lambda}} = \frac{1600\lambda^2 e^{-60\lambda}}{3600\lambda^2 e^{-60\lambda}} = \frac{1600}{3600} = \frac{4}{9}. \]

Thus, the probability that at least one person arrived in the first 20 minutes is

\[ 1 - \frac{4}{9} = \frac{5}{9}. \]

Ch. 9 Theo. ex. p. 484 #2

This means that in \( s \) seconds, there are no cars, with \( \lambda = 3/60 = .05 \):

\[ P(N(s) = 0) = \frac{(0.05s)^0}{0!} e^{-0.05s} = e^{-0.05s}. \]

For the given values
<table>
<thead>
<tr>
<th>$s$</th>
<th>Prob. of surviving</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$e^{-1} = .9048$</td>
</tr>
<tr>
<td>5</td>
<td>$e^{-2.5} = .7788$</td>
</tr>
<tr>
<td>10</td>
<td>$e^{-5} = .6065$</td>
</tr>
<tr>
<td>20</td>
<td>$e^{-1} = .3679$</td>
</tr>
</tbody>
</table>

Ch. 9 Theo. ex. p. 484 #3
In this case, we need $P(N(s) \leq 1)$, which is

$$P(N(s) \leq 1) = \frac{(0.05s)^0}{0!}e^{-0.05s} + \frac{(0.05s)^1}{1!}e^{-0.05s} = (1 + .05s)e^{-0.05s}$$

For the given values,

<table>
<thead>
<tr>
<th>$s$</th>
<th>Prob. of surviving</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$1.25e^{-2.5} = .9735$</td>
</tr>
<tr>
<td>10</td>
<td>$1.5e^{-5} = .9098$</td>
</tr>
<tr>
<td>20</td>
<td>$2e^{-1} = .7358$</td>
</tr>
<tr>
<td>30</td>
<td>$2.5e^{-1.5} = .5579$</td>
</tr>
</tbody>
</table>

Problem 2 Suppose $N(t)$ is a Poisson process with $\lambda = 2$. Find:

a. $P(N(2) = 3)$

$$\frac{(2 \cdot 2)^3}{3!}e^{-2} = .1954$$

b. $P(N(10) - N(7) = 4)$

$$= P(N(3) = 4) = \frac{(2 \cdot 3)^4}{4!}e^{-2} = .1339$$

c. $P(N(10) - N(7) = 4 \mid N(2) = 3)$

$$= P(N(3) = 4) = .1339$$

d. $P(N(5) - N(2) = 4 \mid N(3) = 1)$. **Hint:** Split the event $N(3) = 1$ according to the possible values of $N(2)$.

One approach:

$$= \frac{P(N(5) - N(2) = 4 \cap N(3) = 1)}{P(N(3) = 1)}$$

$$= \frac{P(N(5) - N(2) = 4 \cap N(2) = 1 \cap N(3) - N(2) = 0) + P(N(5) - N(2) = 4 \cap N(2) = 0 \cap N(3) - N(2) = 1)}{P(N(3) = 1)}$$

$$= \frac{P(N(5) - N(3) = 4 \cap N(2) = 1 \cap N(3) - N(2) = 0) + P(N(5) - N(3) = 3 \cap N(2) = 0 \cap N(3) - N(2) = 1)}{P(N(3) = 1)}$$

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\[ P(N(5) - N(3) = 4)P(N(2) = 1)P(N(3) - N(2) = 0) + P(N(5) - N(3) = 3)P(N(2) = 0) \cap P(N(3) - N(2) = 1) \]

\[ = \frac{(2 \cdot 2)^4 e^{-2 \cdot 2} + (2 \cdot 2)^3 e^{-2 \cdot 2}}{4! e^{-2}} \cdot \frac{(2 \cdot 1)^0 e^{-2 \cdot 1} + (2 \cdot 2)^3 e^{-2 \cdot 2}}{3! e^{-2}} \cdot \frac{(2 \cdot 1)^1 e^{-2 \cdot 1}}{1! e^{-2}} \]

\[ = \frac{4^4 \cdot 1 \cdot 3!}{3!} e^{-4 - 4 - 2} + \frac{4^3 \cdot 2 \cdot 3!}{3!} e^{-4 - 4 - 2} \]

\[ = \frac{1024}{24} + \frac{128}{6} e^{-4} \]

\[ = \frac{1536}{144} e^{-4} = .1954 \]

**Another approach:**

\[ P(N(5) - N(2) = 4 | N(3) = 1 \cap N(2) = 0)P(N(2) = 0 | N(3) = 1) + P(N(5) - N(2) = 4 | N(3) = 1 \cap N(2) = 1)P(N(2) = 1 | N(3) = 1) \]

\[ = \frac{4^3 e^{-4} \cdot 1}{3!} + \frac{4^4 e^{-4} \cdot 2}{3!} \]

\[ = .1954 \]

**Problem 3** Let \( W(t) \) be a Brownian process so that \( W(1) \) is a standard normal random variable. Suppose \( S(t) \) is the value of an investment, so that at \( t = 0 \), the value of the investment is $100; and at \( t = 2 \), the value of the investment is normally distributed with a mean of $120 and standard deviation is $40. If

\[ S(t) = a + bt + cW(t) \]

then find \( a \), \( b \), and \( c \).

For \( t = 0 \), \( S(0) = 100 \), so

\[ S(0) = a + b \cdot 0 + c(0) = a = 100 \]

At \( t = 2 \), \( W(2) \) is normal with mean 0 and standard deviation \( \sqrt{2} \); therefore, \( S(2) \) will be normal with mean 100 + 2b and standard deviation \( c\sqrt{2} \). If the mean is 120 and \( \sigma = 40 \), then

\[ 100 + 2b = 120 \]

\[ c\sqrt{2} = 40 \]

so that \( a = 100 \), \( b = 10 \), and \( c = \frac{40}{\sqrt{2}} = 28.284 \)

**Problem 4** For the previous problem, find the value of \( t \) so that we are 99% certain that \( S(t) \geq $150. \)
We start with

\[ P(S(t) \geq 150) = .99 \]

where \( S(t) \) is normal with \( \mu = 100 + 10t \) and \( \sigma = 28.284\sqrt{t} \). The corresponding \( Z \) score is

\[ P(Z \geq -2.33) = .99 \]

so

\[ \frac{150 - (100 + 10t)}{28.284\sqrt{t}} = -2.33 \]

Solve for \( t \):

\[
\begin{align*}
150 - 100 - 10t &= -2.33 \cdot 28.284\sqrt{t} \\
50 - 10t &= -65.9017\sqrt{t} \\
(50 - 10t)^2 &= 4343t \\
2500 - 1000t + 100t^2 &= 4343t \\
100t^2 - 5343t + 2500 &= 0 \\
t &= \frac{5343 \pm \sqrt{5343^2 - 4 \cdot 100 \cdot 2500}}{2 \cdot 100} \\
&= \frac{5343 \pm 5249}{200} \\
&= 52.96 \text{ or } .47
\end{align*}
\]

When we plug these into the original equation, it is apparent that \( t = 52.96 \) is the only solution (\( t = .47 \) gives a \( z \) score of +2.33).

**Problem 5** For the same investment, find \( P(S(5) \geq 160 \mid S(3) = 140) \). *Hint: rewrite \( S(t) \) in terms of a Brownian process starting at \( t = 3 \).*

If we know that \( S(3) = 140 \), then \( S(t) \) becomes a stock model with \( t-3 \), with an additional \( 140-100 = 40 \): \( S(t-3) + 40 \). Therefore we need

\[ P(S(2) + 40 \geq 160) \]

Since \( S(2) + 40 \) is a normal random variable with \( \mu = 2 \cdot 10 + 140 = 160 \) and \( \sigma = 28.284 \cdot \sqrt{2} \), so in terms of \( Z \), we need

\[ P(Z \geq 0) = .5 \]

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