Math 510 HW #21 Selected Answers

Problem 1  Ch. 8 Problems p. 457 #3

We assume \( \bar{X} = \frac{X_1 + \cdots + X_n}{n} \) is normal, due to the Central Limit Theorem. We use \( \mu = E[\bar{X}] = 75 \) and \( \sigma = SD(\bar{X}) = \frac{5}{\sqrt{n}} \). We want

\[ P(70 < X < 80) \geq 0.9 \]

If we balance the probability .9 around the mean, then there is .05 probability to the left of the interval, and .05 probability to the right of it.

According to the normal distribution table, the corresponding \( z \) scores for the probabilities .05 and .95 is:

\[ P(-1.645 < Z < 1.645) = .9 \]

Then

\[ 1.645 = \frac{80 - 75}{\frac{5}{\sqrt{n}}} \]

and solving for \( n \), we have

\[ 1.645 = \sqrt{n} \]

and

\[ n = 1.645^2 = 2.71 \]

so \( n \geq 3 \).

Ch. 8 Problems p. 457 #4a

Now \( X_1 + \cdots + X_{20} \) has mean 20, and is non-negative, so Markov’s inequality says

\[ P(\sum X_i > 15) \leq \frac{20}{15} \]

Ch. 8 Problems p. 457 #4b

We compute that \( \sum_{i=1}^{20} X_i \) has expected value 20 and standard deviation \( \sqrt{20} \).

Using the Central Limit theorem, \( \sum_{i=1}^{20} X_i \) is approximately normal with mean 20 and standard deviation \( \sqrt{20} \).

\[ P(\sum X_i > 15) = P(Z > \frac{15 - 20}{\sqrt{20}}) \]
\[ = P(Z > -1.118) \]
\[ \approx .8686 \]

Ch. 8 Problems p. 457 #5

If \( E_1, E_2, \ldots, E_{50} \) are the roundoff errors for the fifty numbers, the total error is

\[ X = E_1 + \cdots + E_{50} \].

By the central limit theorem this is approximately normal. Since each \( E_i \) has mean 0 and standard deviation \( \frac{1}{\sqrt{12}} \), \( X \) has \( \mu = 0 \) and \( \sigma = \frac{\sqrt{50}}{\sqrt{12}} \).
We want to find

\[ P(X < -3 \text{ or } X > 3) = P(X < -3) + P(X > 3) = 2P(X < -3). \]

We turn this into the standard normal random variable \( Z = \frac{X - \mu}{\sigma} = \frac{X \sqrt{12}}{\sqrt{50}}. \)

\[
2P(X < -3) = 2P\left( Z < -\frac{3\sqrt{12}}{\sqrt{50}} < Z \right)
= 2P(Z < -1.47)
= 2(1 - .9292)
= 2(.0708)
= .1416
\]

**Ch. 8 Problems p. 457 #6**

If \( X_k \) is the result of the \( k \)th role, we want the probability

\[ P(X_1 + \cdots + X_{79} \leq 300) \]

By the central limit theorem, \( X_1 + \cdots + X_{79} \) is approximately normal. Its mean is \( \overline{X} = \frac{7}{2} \cdot 79 = 276.5 \) and standard deviation is \( \sigma = \sqrt{\frac{79}{12}} \sqrt{\frac{35}{72}} = 15.17948. \) So the corresponding probability with \( z \) scores is

\[
P\left( Z \leq \frac{300 - 276.5}{15.17948} \right) = P(Z < 1.548) = .9394
\]

**Ch. 8 Problems p. 457 #7**

Let \( X_k \) be the lifetime of the \( k \)th lightbulb. Since it is exponential with \( E[X_k] = 5 \), we have \( SD(X_k) = 5. \) We want to find

\[ P(X_1 + \cdots + X_{100} > 525). \]

We use the central limit theorem to say that \( X_1 + \cdots + X_{100} \) is approximately normal. Its mean is \( 100 \cdot 5 = 500 \) and its standard deviation is \( \sqrt{100 \cdot 5} = 50. \) Therefore in terms of \( z \) scores we need to find

\[
P\left( Z > \frac{525 - 500}{50} \right) = P(Z > 5) = 1 - .6915 = .3085
\]

**Ch. 8 Problems p. 457 #8**

Let \( Y_k \) be the time it takes to change the \( k \)th lightbulb. This is uniform on \( (0, 5) \), so its mean is 2.5 and standard deviation is \( \frac{\sqrt{5^2}}{12} = .1443. \) We need to find

\[ P(X_1 + Y_1 + \cdots + X_{99} + Y_{99} + X_{100} < 550) \]
This is approximately a normal random variable with mean 100 \cdot 5 + 99 \cdot .25 = 524.75 and standard deviation \sqrt{100 \cdot .25 + 99 \cdot .1443} = 50.02. Thus we need

\[ P(Z < \frac{550 - 524.75}{50.1427}) = P(Z < .505) = .695 \]

**Ch. 8 Problems p. 457 #13a**

Let \( \overline{X} \) be the average test score in the class of size 25. Then \( E[\overline{X}] = 74 \) and \( SD(\overline{X}) = \frac{14}{\sqrt{25}} = 2.8 \).

\[ P(\overline{X} > 80) = P(Z > \frac{80 - 74}{2.8}) = P(Z > 2.143) = 1 - .9838 = .0162 \]

**Ch. 8 Problems p. 457 #13b**

Let \( \overline{Y} \) be the average test score in the class of size 64. Then \( E[\overline{Y}] = 74 \) and \( SD(\overline{Y}) = \frac{14}{\sqrt{64}} = 1.75 \).

\[ P(\overline{Y} > 80) = P(Z > \frac{80 - 74}{1.75}) = P(Z > 3.428) = 1 - .9997 = .0003 \]

**Ch. 8 Problems p. 457 #13c**

Now \( \overline{X} - \overline{Y} \) is normal with mean 0 and standard deviation \( \sqrt{2.8^2 + 1.75^2} = 3.302 \).

\[ P(\overline{X} - \overline{Y} > 2.2) = P(Z > \frac{2.2 - 0}{3.302}) = P(Z > .6663) = 1 - .7486 = .2514 \]

**Ch. 8 Problems p. 457 #13d**

\[ P(\overline{X} - \overline{Y} < -2.2) = P(Z < \frac{-2.2 - 0}{3.302}) = P(Z < -.6663) = 1 - .7486 = .2514 \]

**Problem 2** In this and the next several problems, we will explore why we used \( S_n^4 \) in the proof of the strong law of large numbers.

Why couldn’t we use \( S_n^2 \) in the proof? Work through the proof with \( S_n^2 \) and identify where the proof fails.

As in the original proof, we assume \( E[X_i] = 0 \). Then

\[
E[S_n^2] = E[(X_1 + \cdots + X_n)(X_1 + \cdots + X_n)] = E[X_1^2] + \cdots + E[X_n^2] = nE[X_1^2]
\]

\[
E \left[ \frac{S_n^2}{n^2} \right] = \frac{E[X_1^2]}{n}
\]

\[
\sum_{n=1}^{\infty} E \left[ \frac{S_n^2}{n^2} \right] = \sum_{n=1}^{\infty} \frac{E[X_1^2]}{n}
\]

Since the right side does not converge (the integral test), the proof fails at this point.
Problem 3  Show that it is possible for a sequence $A_n$ of random variables to never converge (i.e., $\lim_{n \to \infty} A_n$ diverges), and yet $\lim_{n \to \infty} E[A_n]$ converges.  

Hint: flip a fair coin ONCE. Define $A_n$ in terms of the outcome of this flip. (Note: $A_n$ does not depend on the $n$th flip of the coin; only the first (and only) flip of the coin.) Make sure $E[A_n] = 0$ for all $n$.  

Flip a fair coin once. Define

\[
A_n = \begin{cases} 
  n, & \text{if the coin is heads} \\
  -n, & \text{if the coin is tails} 
\end{cases}
\]

Then the sequence $\{A_1, A_2, \ldots\}$ has the following probability distribution:

<table>
<thead>
<tr>
<th>outcome</th>
<th>${A_1, A_2, A_3, \ldots}$</th>
<th>converges?</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>${1, 2, 3, \ldots}$</td>
<td>diverges</td>
</tr>
<tr>
<td>T</td>
<td>${-1, -2, -3, \ldots}$</td>
<td>diverges</td>
</tr>
</tbody>
</table>

Therefore, $\{A_n\}$ always diverges. But $E[A_n] = 0$, so $\lim_{n \to \infty} E[A_n] = 0$.  

Problem 4  We proved in class that if $A_n$ is an increasing sequence of random variables (that is, in every case, $A_{n+1} \geq A_n$), then $\lim_{n \to \infty} E[A_n]$ converging implies $\lim_{n \to \infty} A_n$ converges with probability 1.  

How does this relate to not choosing $S^3_n$ in the proof of the strong law of large numbers?  

For $S^3_n$ we would have

\[
E \left[ \sum_{n=1}^{\infty} \frac{S^3_n}{n^3} \right] \text{ converges}
\]

but since the terms $\frac{S^3_n}{n^3}$ are not necessarily positive, we cannot know if $\sum_{n=1}^{N} \frac{S^3_n}{n^3}$ increases as $N$ increases. So we cannot conclude that with probability 1, $\frac{S^3_n}{n^3}$ converges.  

Problem 5  In general, which values of $k$ could we use in $S^k_n$ in the proof of the strong law of large numbers?  

Any $k$ which is even and $> 2$. This explains why $k = 4$ was used in the book’s proof.