Math 510 HW #19 selected answers

Problem 1 Let $X$ be a random variable with the following mass distribution:

<table>
<thead>
<tr>
<th>$X$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.3</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Find the moment generating function for $X$.

$$M_X(t) = E[e^{tX}]$$
$$= 0.3 e^{0t} + 0.5 e^{1t} + 0.2 e^{2t}$$
$$= 0.3 + 0.5 e^{t} + 0.2 e^{2t}$$

Problem 2 Let $X$ be a random variable with the density function

$$f(x) = \begin{cases} 
    x, & 0 < x < 1 \\
    0, & \text{otherwise}
\end{cases}$$

Find the moment generating function for $X$.

$$M(t) = E[e^{tX}]$$
$$= \int_{-\infty}^{\infty} f(x) e^{tx} dx$$
$$= \int_{0}^{1} x e^{tx} dx$$
$$= \frac{x}{t} e^{tx} \bigg|_{x=1} - \int_{0}^{1} \frac{1}{t} e^{tx} dx$$
$$= \frac{1}{t} e^{t} - \frac{1}{t^2} (e^{t} - 1)$$

Problem 3 Suppose you have a random variable $X$ whose moments are given by $E[X^n] = n!$. Find the moment generating function for $X$.

$$M(t) = \sum_{n=0}^{\infty} \frac{E[X^n]}{n!} t^n$$
$$= \sum_{n=0}^{\infty} \frac{n!}{n!} t^n$$
$$= \sum_{n=0}^{\infty} t^n$$
$$= \frac{1}{1 - t}$$
**Problem 4** Prove that the sum of two independent Poisson random variables with $\lambda = 1$ is a Poisson random variable with $\lambda = 2$ using moment generating functions.

The moment generating function for a Poisson random variable with $\lambda = 1$ is $M_X(t) = e^{e^t-1}$. The sum of two independent such is $M_X(t)^2 = e^{2(e^t-1)}$, which is the moment generating function for a Poisson random variable with $\lambda = 2$. By the uniqueness of moment generating functions, the sum of two independent Poisson random variables with $\lambda = 1$ is a Poisson random variable with $\lambda = 2$.

**Problem 5** Find the first four moments (i.e., $E[X]$ through $E[X^4]$) of an exponential distribution with parameter $\lambda$, by taking derivatives of the moment generating function.

The moment generating function for an exponential random variable is

$$M(t) = \frac{\lambda}{\lambda - t}$$

The first four derivatives are:

$$M'(t) = \frac{\lambda}{(\lambda - t)^2}$$

$$M''(t) = \frac{2\lambda}{(\lambda - t)^3}$$

$$M'''(t) = \frac{6\lambda}{(\lambda - t)^4}$$

$$M''''(t) = \frac{24\lambda}{(\lambda - t)^5}$$

and to find the moments, we plug in $t = 0$:

$$E[X] = M'(0) = \frac{1}{\lambda}$$

$$E[X^2] = M''(0) = \frac{2}{\lambda^2}$$

$$E[X^3] = M'''(0) = \frac{6}{\lambda^3}$$

$$E[X^4] = M''''(0) = \frac{24}{\lambda^4}$$

**Problem 6** Suppose the moment generating function for $X$ is

$$M(t) = e^{4t^2+3t}.$$ Describe what kind of random variable $X$ is, with what values of the parameters.
By the uniqueness of moment generating functions, $X$ must be normal, since normal random variables have moment generating functions

$$e^{\frac{\sigma^2}{2}t^2 + \mu t}.$$ 

Matching terms, we conclude $\sigma^2/2 = 4$ so that $\sigma = \sqrt{8}$, and $\mu = 3$.

**Problem 7 Theoretical ex. #46:**

We expand the moment generating function for $Z$:

$$M_Z(t) = e^{t^2/2} = \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{t^2}{2} \right)^j = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{t^{2j}}{2^j}.$$

The $t^n$ term is 0 if $n$ is odd, and when $n$ is even, we write $n = 2j$ and the term is

$$\frac{1}{2^j j!} t^n.$$

Comparing this with $\frac{\mu_n}{n!} t^n$, we get that when $n = 2j$,

$$\mu_n = \frac{(2j)!}{2^j j!}.$$

**Theoretical ex. #49:**

The fact that $\log X$ is normal means

$$E[e^{t \log X}] = e^{\frac{\sigma^2}{2}t^2 + \mu t}$$

and

$$E[e^{t \log X}] = E[X^t]$$

so setting $t = 1$ and $t = 2$ we get

$$E[X] = E[X^1] = e^{\frac{\sigma^2}{2} + \mu}$$

and

$$E[X^2] = e^{\frac{3\sigma^2}{2} + 2\mu}$$

so that

$$Var(X) = E[X^2] - E[X]^2 = e^{2\sigma^2 + 2\mu} - e^{\sigma^2 + 2\mu}$$

**Theoretical ex. #50:**
\[ \Psi(t) = \log M(t) \]

\[ \Psi'(t) = \frac{M'(t)}{M(t)} \]

\[ \Psi''(t) = \frac{M''(t) M(t) - M'(t) \cdot M'(t)}{M(t)^2} \]

\[ \Psi''(0) = \frac{M''(0) M(0) - M'(0) \cdot M'(0)}{M(0)^2} \]

\[ = \frac{E[X]^2 \cdot 1 - E[X]^2}{12} \]

\[ = E[X]^2 - E[X]^2 \]

\[ = \operatorname{Var}(X) \]