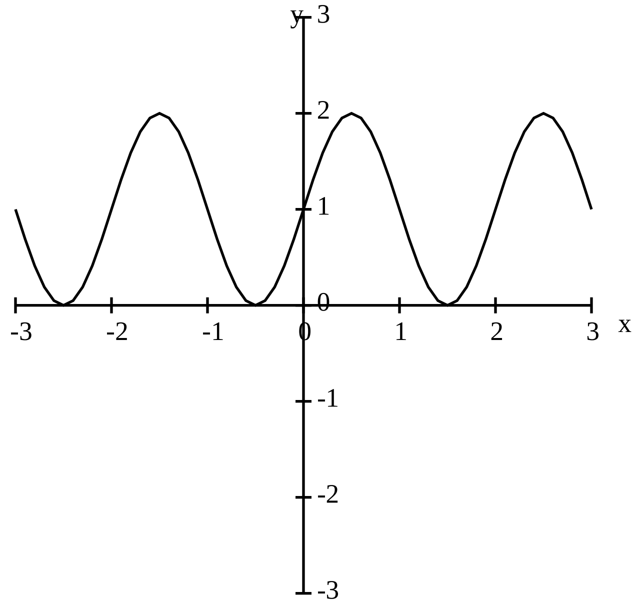
Math 104 Midterm 2 answers

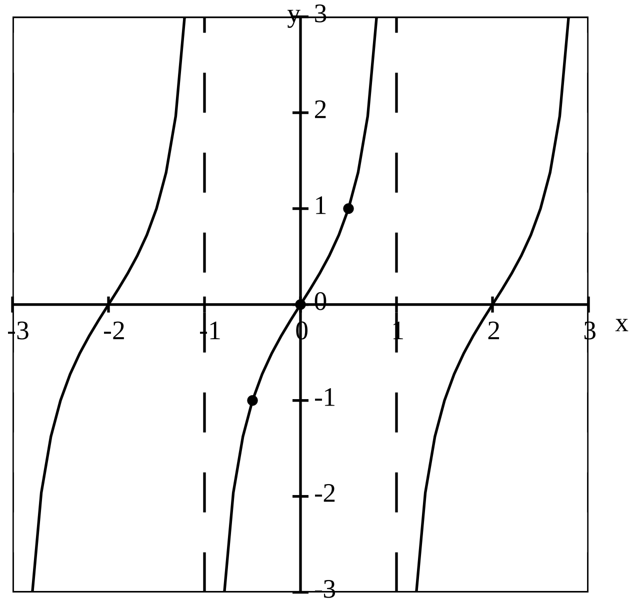
Problem 1 (5 pts) *Sketch the graph of* $y = \sin(\pi x) + 1$ *in the graph given below.*

We establish the period as $(2\pi)/(\pi) = 2$. *Other than that the graph is shifted up by 1. Note in this and other graphs that you were supposed to graph the function in the domain given, not just give one period.*



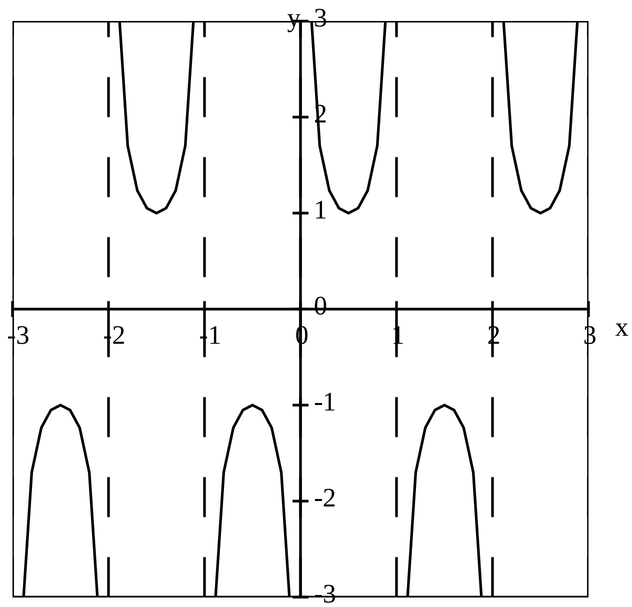
Problem 2 (5 pts) *Sketch the graph of* $y = \tan(\frac{\pi}{2}x)$ *in the graph given below.*

The period is $\pi/(\pi/2) = 2$, *and the first asymptote occurs when* $x = 1$, *because that is when we plug* $\pi/2$ *into* tan.

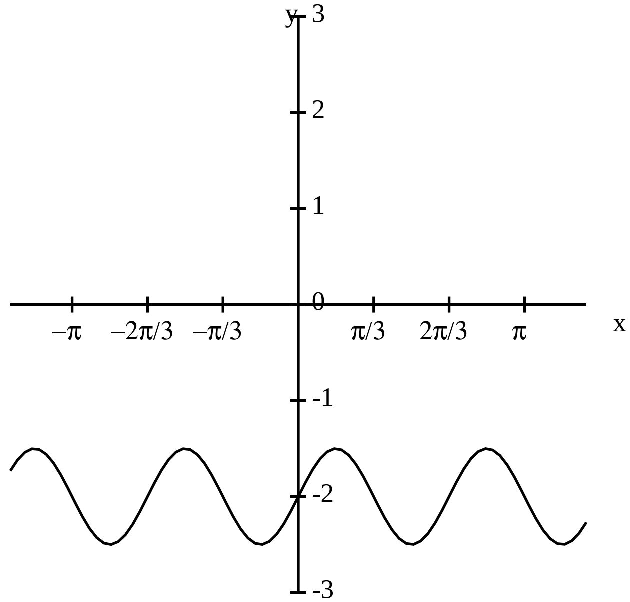


Problem 3 (5 pts) *Sketch the graph of* $y = \csc(\pi x)$ *in the graph given below.*

The period is $(2\pi)/\pi = 2$*, and mimics* sin*, except going to* ∞ *when* sin *goes to 0. Otherwise, it has the same sign as* sin*.*



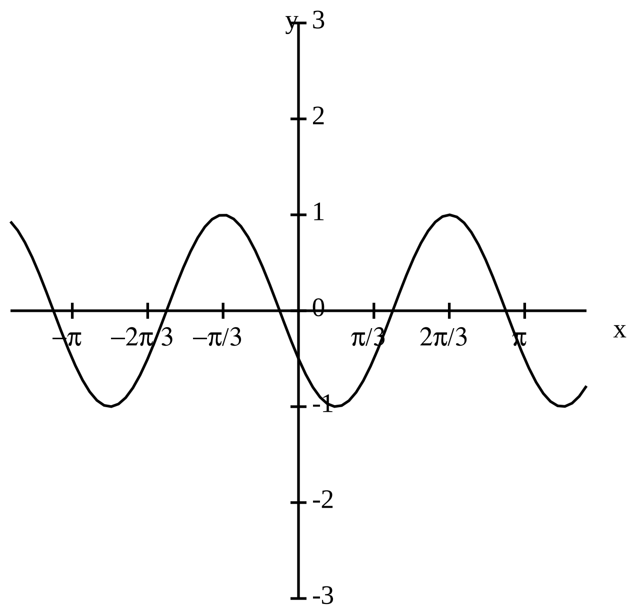
Problem 4 (5 pts) *Below is a graph. What is its formula?*



This is a sine graph, with amplitude 1/2 and shifted down by 2 units, of period $2\pi/3$. The b value is $(2\pi)/(2\pi/3) = 3$. Thus this is a graph of

$$\frac{1}{2}\sin(3x) - 2$$

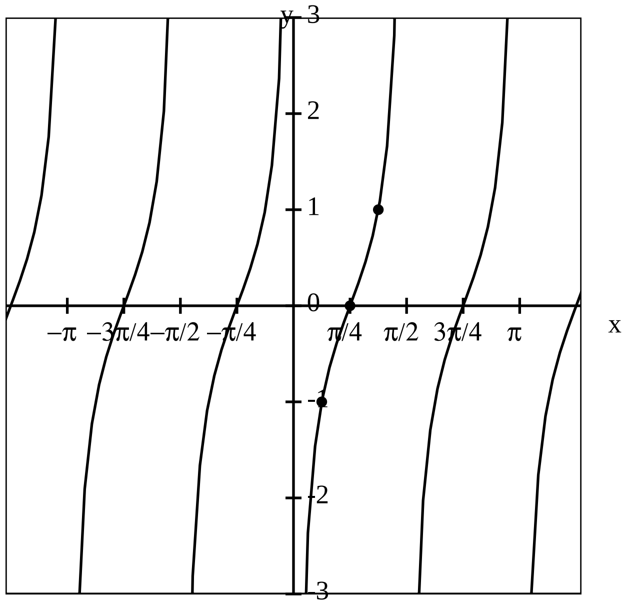
Problem 5 (5 pts) *Below is a graph. What is its formula?*



This is like a cosine graph, shifted by $\pi/3$ to the left. The period can be obtained by identifying two crests, say at $-\pi/3$ and $2\pi/3$, and noting that they are $2\pi/3 - (-\pi/3) = \pi$ apart. Thus the period is π and the b value is $(2\pi)/\pi = 2$. We have

$$\cos(2(x + \frac{\pi}{3}))$$

Problem 6 (5 pts) *Below is a graph. What is its formula? Note that the half and quarter cycle points are indicated by dots on the graph.*



This looks like a tangent graph, but it has asymptotes at $x = 0$ and $x = \pi/2$, indicating a period of $\pi/2$ instead of π, resulting in $\tan(2x)$. There is no vertical stretching because the quarter points are at ± 1 as they should be. But we need to shift it $\pi/4$ to the left or right. When we do this to $\tan(2x)$ we get:

$$\tan(2(x - \frac{\pi}{4}))$$

Note that there are other acceptable answers to this problem.

Problem 7 (5 pts) *If we have a function $f(x)$, what is the formula for the function whose graph is just like $f(x)$, but shifted 8 units to the left and 3 units up?*

If $f(x)$ is the original function, we shift left by 8 units by writing $f(x + 8)$ and we shift up by adding 3 after performing the function, so altogether, the function is:

$$f(x + 8) + 3$$

Problem 8 (5 pts) *For the following function, give the amplitude and period. Your expression may involve π.*

$$y = 35 \sin(70x + 5) + 300$$

The amplitude is 35, and the period is $(2\pi)/70 = \pi/35$.

Problem 9 (5 pts) *Use trigonometric identities to simplify as much as possible:*

$$\sec\theta\cos\theta$$

$$\begin{aligned}\sec\theta\cos\theta &= \frac{1}{\cos\theta}\cos\theta \\ &= 1\end{aligned}$$

Problem 10 (5 pts) *Use trigonometric identities to simplify as much as possible:*

$$\csc x\tan x$$

$$\begin{aligned}\csc x\tan x &= \frac{1}{\sin x}\frac{\sin x}{\cos x} \\ &= \frac{1}{\cos x} \\ &= \sec x\end{aligned}$$

Problem 11 (5 pts) *Use trigonometric identities to simplify as much as possible:*

$$\frac{\sin x\tan x + \cos x}{\tan x}$$

$$\begin{aligned}\frac{\sin x\tan x + \cos x}{\tan x} &= \frac{\sin x\frac{\sin x}{\cos x} + \cos x}{\frac{\sin x}{\cos x}} \\ &= \frac{\frac{\sin x\sin x}{\cos x} + \frac{\cos x\cos x}{\cos x}}{\frac{\sin x}{\cos x}} \\ &= \frac{\frac{\sin^2 x+\cos^2 x}{\cos x}}{\frac{\sin x}{\cos x}} \\ &= \frac{\sin^2 x + \cos^2 x}{\cos x}\cdot\frac{\cos x}{\sin x} \\ &= \frac{1}{\cos x}\cdot\frac{\cos x}{\sin x} \\ &= \frac{1}{\sin x} \\ &= \csc x\end{aligned}$$

Problem 12 (5 pts) *Use trigonometric identities to simplify as much as possible:*

$$\sin(2x)\sec^2 x$$

$$\begin{aligned}
\sin(2x)\sec^2 x &= 2\sin x\cos x\sec^2 x \\
&= 2\sin x\cos x\frac{1}{\cos^2 x} \\
&= 2\sin x\frac{1}{\cos x} \\
&= 2\tan x
\end{aligned}$$

Problem 13 (5 pts) *Use trigonometric identities to simplify as much as possible:*

$$\frac{\sin(x+y)+\sin(x-y)}{2}$$

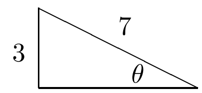
$$\begin{aligned}
\frac{\sin(x+y)+\sin(x-y)}{2} &= \frac{(\sin x\cos y+\cos x\sin y)+(\sin x\cos y-\cos x\sin y)}{2} \\
&= \frac{\sin x\cos y+\cos x\sin y+\sin x\cos y-\cos x\sin y}{2} \\
&= \frac{2\sin x\cos y}{2} \\
&= \sin x\cos y
\end{aligned}$$

Problem 14 (5 pts) *If* $\tan x = 2$ *and* $\tan y = 3$, *find* $\tan(x+y)$.

$$\begin{aligned}
\tan(x+y) &= \frac{\tan x+\tan y}{1-\tan x\tan y} \\
&= \frac{2+3}{1-2\cdot 3} \\
&= \frac{5}{1-6} \\
&= \frac{5}{-5} \\
&= -1
\end{aligned}$$

Problem 15 (5 pts) *If* $\sin\theta = 3/7$, *and* θ *is in the second quadrant, find* $\cos\theta$.

We draw a triangle with the angle θ shown in the second quadrant:



The missing leg is

$$\sqrt{7^2 - 3^2} = \sqrt{49 - 9} = \sqrt{40}.$$

The cosine of θ is the adjacent over hypotenuse, or

$$\frac{\sqrt{40}}{7}$$

but in the second quadrant, this is negative, so we have

$$\cos\theta = \frac{-\sqrt{40}}{7}$$

Problem 16 (5 pts) *Using θ from the previous problem, find* $\tan(\theta/2)$.

We calculate:

$$\begin{aligned}\tan(\theta/2) &= \frac{1-\cos\theta}{\sin\theta}\\ &= \frac{1 - \frac{-\sqrt{40}}{7}}{\frac{3}{7}}\\ &= \frac{\frac{7}{7} + \frac{\sqrt{40}}{7}}{\frac{3}{7}}\\ &= \left(\frac{7+\sqrt{40}}{7}\right)\left(\frac{7}{3}\right)\\ &= \frac{7+\sqrt{40}}{3}\end{aligned}$$

Note that the sign of the answer in the previous problem is relevant to getting this answer. Also note that though θ is in the second quadrant, since our answer to this problem is positive, apparently $\theta/2$ is in the first quadrant.

Problem 17 (5 pts) *If θ is an angle for which $\cos\theta = 2/5$, find the possible values of* $\sin(\theta/2)$.

$$\begin{aligned}
\sin(\theta/2) &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\
&= \pm\sqrt{\frac{1-\frac{2}{5}}{2}} \\
&= \pm\sqrt{\frac{\frac{3}{5}}{2}} \\
&= \pm\sqrt{\frac{3}{5}\cdot\frac{1}{2}} \\
&= \pm\sqrt{\frac{3}{10}}
\end{aligned}$$

Note that both of these answers are possible, because adding 2π to θ will add π to $\theta/2$, and change the sign of $\sin(\theta/2)$.

Problem 18 (5 pts) *Given that* $\cos 25° = 0.906$*, find* $\cos^{-1}(-0.906)$ *in degrees.*

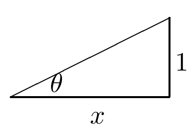
We need to find an angle θ with $\cos\theta = -0.906$. Since $\cos^{-1}$ takes values from $0°$ to $180°$, we must look at the second quadrant to find negative values. We know $\cos 25° = 0.906$, and $180° - 25°$ has the same reference angle, with the opposite sign for cos, so the answer must be $180° - 25° = 155°$.

Problem 19 (5 pts) *Write an algebraic formula (that is, one that does not involve trigonometric functions) for*

$$\sin(\cot^{-1}(x)).$$

Be sure to explain how you choose the correct sign.

Let $\theta = \cot^{-1}(x)$. We begin by drawing a triangle:



and label the 1 and x accordingly since $\cot\theta = x$.

The missing length is $\sqrt{x^2+1^2} = \sqrt{x^2+1}$. To find $\sin\theta$ we look at the opposite over the hypotenuse, which gives us

$$\frac{1}{\sqrt{x^2+1}}.$$

Now we check the sign. If x is positive, $\theta = \cot^{-1} x$ is in the first quadrant, and $\sin\theta$ should be positive. If x is negative, $\cot^{-1} x$ is in the second quadrant, and $\sin\theta$ should still be positive. The formula we have is always positive, so we know we have the correct sign. The answer is thus

$$\frac{1}{\sqrt{x^2+1}}.$$

Problem 20 (5 pts) *Verify that the following is an identity:*

$$\frac{\sin\theta}{1-\cos\theta} - \frac{\sin\theta\cos\theta}{1+\cos\theta} = \csc\theta(1+\cos^2\theta)$$

We start on the left side, putting everything over one common denominator:

$$\begin{aligned}
\frac{\sin\theta}{1-\cos\theta} - \frac{\sin\theta\cos\theta}{1+\cos\theta} &= \frac{\sin\theta}{1-\cos\theta}\frac{1+\cos\theta}{1+\cos\theta} - \frac{\sin\theta\cos\theta}{1+\cos\theta}\frac{1-\cos\theta}{1-\cos\theta} \\
&= \frac{\sin\theta(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)} - \frac{\sin\theta\cos\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)} \\
&= \frac{\sin\theta(1+\cos\theta) - \sin\theta\cos\theta(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)} \\
&= \frac{\sin\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + \sin\theta\cos^2\theta}{1-\cos^2\theta} \\
&= \frac{\sin\theta + \sin\theta\cos^2\theta}{\sin^2\theta} \\
&= \frac{\sin\theta(1+\cos^2\theta)}{\sin^2\theta} \\
&= \frac{1+\cos^2\theta}{\sin\theta} \\
&= \frac{1}{\sin\theta}\cdot(1+\cos^2\theta) \\
&= \csc\theta\cdot(1+\cos^2\theta)
\end{aligned}$$

which is what we have on the right side.