1 Abstract for Salt Lake City Talk

In anisotropic diffusion we time-march
\[ u_t = \nabla \cdot (g(|\nabla u|) \nabla u) \tag{1} \]
with \( u(\vec{x}, t = 0) = u_0(\vec{x}) \). One way to solve the TV Regularization problem
\[ \min_u \frac{1}{2} \| u - u_0 \|^2 + \alpha TV(u). \tag{2} \]
is to time-march
\[ u_t = \alpha \nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right) - (u - u_0) \]
until \( u_t = 0 \), i.e.,
\[ u = u_0 + \alpha \nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right). \tag{3} \]
Thus solving (2) is equivalent to performing a single \emph{implicit} time-march step, using (1) where \( g(|\nabla u|) = 1/|\nabla u| \), with step size \( \alpha \), and the result of \emph{explicitly} time-marching (1) \( n \) steps, where \( n \Delta t = \alpha \), will be approximately the same. We present current results of our on-going investigation of this relationship and discuss its usefulness.

2 Questions, Thoughts and Issues

One other equation I reference in this little summary:
\[ u_t = \nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right) \tag{4} \]
with \( u(\vec{x}, t = 0) = u_0(\vec{x}) \).

- Some underlying theory: since we understand how the choice of \( \alpha \) influences what TV Regularization does to an image (this is actually discussed in the paper I sent along with this email), we can choose some \( \alpha \) and time-march \( n \) steps of size \( \Delta t \), where \( n \Delta t = \alpha \).

A second way of looking at it is that if we know the step size \( \Delta t \) and the number of steps \( n \), then the result of solving (4) corresponds to (but is not exactly equal to) the result of solving (2) where \( \alpha = n \Delta t \).

We could also do something with varying step sizes for \( \Delta t \), so long as the sum of the steps is \( \alpha \).
Why consider this problem? One reason is that it's interesting. What are the applications or usefulness of any results we come up with?

- **Motivation**
  A good initial guess or estimate for the solution to (2) would result in fewer iterations being need to reach the final solution, regardless of the numerical scheme being used in solving (2).

- **Approach**
  First thought: an approach that is extreme and not actually mathematically sound (I would assume—I haven't actually done any work to determine this): do one giant explicit time step of (4), using $\Delta t = \alpha$, to get us close to the true solution to (2), and then use this as a starting point for solving (2) using whatever numerical scheme is desired (e.g. whatever is fastest).

  Second thought: for stability, there will be a condition on the maximum size of $\Delta t$, so instead of $\Delta t = \alpha$, make $\Delta t$ this maximumally allowed size, and then do $n$ steps where $n\Delta t = \alpha$. If $n$ is not too large, we could quickly get a result that is potentially a pretty good initial guess for whatever numerical scheme we use.

  Third thought: even if allowed, perhaps this "maximum" value of $\Delta t$ is too large to end up with a result (i.e. a starting value for another numerical scheme) that is of any real value, so choose $\Delta t$ to be smaller.

- Perhaps the results we find for the TV problem could be applied to the variations of TV Regularization that have been developed over the past few years and/or to the various versions of anisotropic diffusion.

- I expect that there are other uses.

- **Experimentation/numerical results:** We could easily enough experiment by choosing a variety of sizes for $\Delta t$ when explicitly time marching (4)—in each case the number of steps $n$ will be such that $n\Delta t = \alpha$—and looking (both in terms of the look of the image, such as smoothing of edges, as well as the numerical difference of the final result after explicitly time marching (4) compared to the true solution to (2)).
Theory: As $\Delta t \to 0$, will the result of explicit time marching (4) always approach the solution to (2)? If not, are there certain conditions under which this would be the case? If not, can we find some way of describing qualitatively and/or quantitatively how close the two solutions would be and/or estimate some bound on their difference?