Exam 2 Solutions  Math 215 Fall 2005

High: 100    Average: 76    Low: 45
If I were to curve this exam: 85 – 100 A; 70 – 85 B; 0 – 70 C or below.

1. Do Gaussian elimination to make the transformation \([A | I] \rightarrow [I | A^{-1}]\):

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix}
\]

so \(A^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix}\). You can check that \(A^{-1}A = I\).

2.

<table>
<thead>
<tr>
<th>Final matrix</th>
<th>Solution</th>
</tr>
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</table>
| \[ \begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 1
\end{bmatrix} \] | \(x = 2\)  
\(y = 1\) |
| \[ \begin{bmatrix}
1 & 0 & 2 \\
0 & 0 & 1
\end{bmatrix} \] | No solution, due to the bottom row. |
| \[ \begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 0
\end{bmatrix} \] | \(x = 2\)  
\(y = 0\) |
| \[ \begin{bmatrix}
1 & -1 & 3 & 6 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \] | \(x = y - 3z + 6\)  
\(y = \text{anything}\)  
\(z = \text{anything}\) |
| \[ \begin{bmatrix}
1 & 0 & 6 \\
0 & 1 & 3 \\
0 & 0 & 0
\end{bmatrix} \] | \(x = 6\)  
\(y = 3\)  
(there is no \(z\): there are three equations and two unknowns) |
3. First,

\[ I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .2 & .4 \\ .4 & .3 \end{bmatrix} = \begin{bmatrix} .8 & -.4 \\ -.4 & .7 \end{bmatrix}, \]

and since the determinant of \( I - A \) is \( \Delta = (.8)(.7) - (-.4)(-.4) = .56 - .16 = .4 \), then

\[ (I - A)^{-1} = \frac{1}{.4} \begin{bmatrix} .7 & .4 \\ .4 & .8 \end{bmatrix} = \begin{bmatrix} 1.75 & 1 \\ 1 & 2 \end{bmatrix}, \]

so the amount to produce is

\[ X = (I - A)^{-1} D = \begin{bmatrix} 1.75 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 27.5 \\ 30 \end{bmatrix}. \]

4. (a) Call the two items Item A and Item B. The first row of numbers means that .2 unit of Item A is used up in producing 1 unit of Item A and .8 unit of Item A is used up in producing 1 unit of Item B. So producing 10 units each of Items A and B uses up, respectively, 2 units and 8 units of Item A, that is, all 10 units of Item A.

The second row similarly says that all of Item B is used up in the production process.

In other words, if the amount produced is \( X = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \), then the amount used up in production is \( AX = \begin{bmatrix} .2 & .8 \\ .7 & .3 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \), precisely the amount produced, so there is none left over.

(b) If \( A = \begin{bmatrix} .2 & .8 \\ .7 & .3 \end{bmatrix} \), then \( I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} .2 & .8 \\ .7 & .3 \end{bmatrix} = \begin{bmatrix} .8 & -.8 \\ -.7 & .7 \end{bmatrix} \), which has no inverse, since its determinant is \( \Delta = (.8)(.7) - (-.7)(-.8) = 0 \), so there is no solution to the problem of solving for \( X \) in \( (I - A)X = D \).

(c) In general, if \( A = \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix} \), then \( I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 - a & a \\ b & 1 - b \end{bmatrix} = \begin{bmatrix} a & -a \\ b & -b \end{bmatrix} \), which has determinant \( \Delta = a(-b) - b(-a) = 0 \), so \( I - A \) has no inverse, which again means that there is no solution to the problem of solving for \( X \) in \( (I - A)X = D \).
5. Same number of equations as unknowns: One solution  

More equations than unknowns: No solution  

Fewer equations than unknowns: Infinite solutions

6. No solution: the two lines are parallel, e.g.  
\[ x + y = 1 \]
\[ x + y = 2 \]

(Note: \(0x + 0y = 1\) or anything similar is not an equation, since \(0 \neq 1\).)

One solution: the two lines are not parallel, e.g.  
\[ x + y = 1 \]
\[ x - y = 2 \]

Infinite solutions: the two lines are actually the same, e.g.  
\[ \frac{1}{2}x + y = 1 \]
\[ 2x + 2y = 2 \]

7. Let \(x\), \(y\) and \(z\) be the amounts to invest in bonds, CDs and mortgages, respectively. Then there are three conditions given in the problem that lead to three equations:
\[ x + y + z = 100000 \]
\[ .05x + .07y + .08z = 6000 \]
\[ x + y - 4z = 0 \]

(The third equation was first thought of and written as \(x + y = 4z\).)

\[
\begin{bmatrix}
1 & 1 & 1 & 100000 \\
.05 & .07 & .08 & 6000 \\
1 & 1 & -4 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 1 & 100000 \\
0 & 7 & 8 & 600000 \\
0 & 0 & -4 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 1 & 100000 \\
0 & 1 & 2 & 3 & 100000 \\
0 & 0 & -5 & -100000 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -\frac{1}{2} & 50000 \\
0 & 1 & 3 & 2 & 500000 \\
0 & 0 & -5 & -100000 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 60000 \\
0 & 1 & 0 & 20000 \\
0 & 0 & 1 & 20000 \\
\end{bmatrix}
\]

So we invest $60,000, $20,000 and $20,000 in bonds, CDs and mortgages, respectively.

If you got the third equation mixed up, then you would have had \(4x + 4y = z\), i.e. \(4x + 4y - z\), and the solution to
\[ x + y + z = 100000 \]
\[ .05x + .07y + .08z = 6000 \]
\[ 4x + 4y - z = 0 \]

would be $90,000, -$70,000 and $80,000. Hopefully the -$70,000 would have caused you to think something was wrong.